Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2013,**33B**(4):1024–1030
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

BLOW-UP OF THE SOLUTION FOR A CLASS OF POROUS MEDIUM EQUATION WITH POSITIVE INITIAL ENERGY*

Xiulan WU (吴秀兰)[1,2] *Wenjie GAO* (高文杰)[1†]
1. Institute of Mathematics, Jilin University, Changchun 130012, China
2. College of Mathematics, Jilin Normal University, Siping 136000, China
E-mail: chjlsywxl@126.com; wjgao@jlu.edu.cn

Abstract This paper deals with a class of porous medium equation

$$u_t = \Delta u^m + f(u)$$

with homogeneous Dirichlet boundary conditions. The blow-up criteria is established by using the method of energy under the suitable condition on the function $f(u)$.

Key words porous medium equation; blow-up; positive initial energy

2010 MR Subject Classification 35K55; 35K65

1 Introduction

In this paper, we consider the following porous equation with sources

$$\begin{cases} u_t = \Delta u^m + f(u), \ (x,t) \in \Omega \times [0,T); \\ u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T); \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^N, $N > 2$, with a smooth boundary $\partial\Omega$, $m > 1$ and $f(u)$ is a continuous function satisfying some conditions to be given later, $u_0(x)$ is a nonnegative function and $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. Problem (1.1) arises from nonlinear fluid dynamics, see [1]. When $m = 1$, the blow-up properties of the semi-linear heat equation (1.1) were investigated by many researchers, see the survey [2]. The cases of fast diffusion were extensively studied for (1.1), we refer the readers to [3–7].

The problems on blow-up to nonlinear parabolic equations were intensively studied (see [8, 16]). The works mentioned above, the authors discussed Fujita exponents to ensure the

*Received May 14, 2012. The project is supported by NSFC (11271154), Key Lab of Symbolic Computation and Knowledge Engineering of Ministry of Education and by the 985 Program of Jilin University.

†Corresponding author: Wenjie GAO.

properties of blowing up by applying upper-lower solutions. To the best of our knowledge, there are a fewer works deals with blow-up solutions when the initial energy is positive. We can refer to [9–11]. Motivated by the above works, in this paper we establish a blow-up result for certain solution with positive initial energy. For the sake of simplicity, we assume that

$$\inf\left\{\int_\Omega F(u)\mathrm{d}x : |u|=1\right\}>0, \tag{1.2}$$

where $F(u)=\int_0^u ms^{m-1}f(s)\mathrm{d}s$ and B is the optimal constants of the embedding inequality

$$\left(\int_\Omega rF(u)\mathrm{d}x\right)^{\frac{1}{r}}\le B\|\nabla u^m\|_2, u^m\in H_0^1(\Omega). \tag{1.3}$$

That is

$$B^{-1}=\inf_{u^m\in H_0^1(\Omega),u\neq 0}\frac{\|\nabla u^m\|_2}{(\int_\Omega rF(u)\mathrm{d}x)^{\frac{1}{r}}},$$

where $r\in(2,\frac{2N}{N-2}]$ is a fixed positive constant. In this paper, the norm $\|\cdot\|_p$ denotes $\|\cdot\|_{L^p(\Omega)}$. Let

$$\alpha_1=B^{-\frac{r}{r-2}},\qquad E_1=\left(\frac{1}{2}-\frac{1}{r}\right)B^{-\frac{2r}{r-2}}, \tag{1.4}$$

and

$$E(t)=\frac{1}{2}\int_\Omega|\nabla u^m|^2\mathrm{d}x-\int_\Omega F(u)\mathrm{d}x. \tag{1.5}$$

It is easy to verify that the following conclusion holds

$$E'(t)=-\int_\Omega mu^{m-1}u_t^2\mathrm{d}x=-\frac{4m}{(m+1)^2}\int_\Omega(u^{\frac{m+1}{2}})_t^2,\quad t>0. \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2, we give the definition of weak solutions to problem (1.1) and some preliminaries. The proofs of the main results will be presented in Section 3.

2 Preliminaries

It is well known that the equation in (1.1) is degenerate if $m>1$, and therefore there is no classical solution in general. We begin with the definition of a weak solution of (1.1).

Definition 2.1 A function u with $u^m\in L^\infty(\Omega\times(0,T))\cap L^2(0,T;H_0^1(\Omega)),(u^{\frac{m+1}{2}})_t\in L^2(\Omega\times(0,T))$ is called a solution of problem (1.1) in Q_T, if the following holds

$$\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x+\int\int_{Q_T}[u\varphi_t-\nabla u^m\cdot\nabla\varphi+f\varphi]\mathrm{d}x\mathrm{d}t=0 \tag{2.1}$$

for any $\varphi\in\Phi$, and u satisfies the initial condition $u(x,0)=u_0(x)\in L^\infty(\Omega)$, where

$$\Phi=\{\varphi|\varphi\in H^1(Q_T),\varphi(x,T)=0,\varphi(x,t)|_{\partial\Omega}=0\}.$$

We have the following lemma with a similar method in [12].

Lemma 2.1 Let $h(s)\in C^1(R),f(s)\in C(R)$ satisfy

$$h(s)>0,|ms^{m-1}f(s)|\le h(s^m), \tag{2.2}$$

then for any $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, there exists a $T' \in (0, T)$ such that problem (1.1) has a solution u with

$$u^m \in L^\infty(\Omega \times (0,T')) \cap L^2((0,T');\ H_0^1(\Omega)),\ (u^{\frac{m+1}{2}})_t \in L^2(\Omega \times (0,T')).$$

Proof We consider the following regularization problem

$$u_t = \triangle u^m + f(u), x \in \Omega, 0 < t < T, \tag{2.3}$$

$$u(x,t) = \varepsilon, x \in \partial\Omega, 0 < t < T, \tag{2.4}$$

$$u(x,0) = u_0(x) + \varepsilon, x \in \Omega, \tag{2.5}$$

where $0 < \varepsilon < 1$, $u_{0\varepsilon}(x)$ satisfies

$$|(u_{0\varepsilon} + \varepsilon)^m|_{L^\infty(\Omega)} \le |(u_0(x) + 1)^m|_{L^\infty(\Omega)},$$

$$|\nabla u_{0\varepsilon}^m|_{L^2(\Omega)} \le |\nabla u_0^m|_{L^2(\Omega)},$$
$$(u_{0\varepsilon})^m \to u_0^m \quad \text{in} \quad H^1(\Omega).$$

By [13] we know that problem (2.3)–(2.5) has a classical solution $u_\varepsilon(x,t)$ and $u_\varepsilon(x,t) \ge \varepsilon$ in $\Omega \times [0, T)$.

First, we can claim that there exists a $T' \in (0, T)$ and a constant M such that

$$|u_\varepsilon^m|_{L^\infty(\Omega\times(0,T'))} \le M \quad \text{for all} \quad 0 < \varepsilon < 1. \tag{2.6}$$

To prove this, let $w(t)$ be the solution of the ordinary differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = h(w), \tag{2.7}$$

$$w(0) = |(u_0(x) + 1)^m|_{L^\infty(\Omega)}. \tag{2.8}$$

By standard theory, Chapter one in [14], there exists a $T_1 \in (0, T)$, which depends on the initial value $|(u_0(x) + 1)^m|_{L^\infty(\Omega)}$, such that the problem (2.7) and (2.8) has a solution w on $[0, T_1]$. Let $\phi(x,t) = u_\varepsilon^m - w$, by (2.2) it follows that

$$m u_\varepsilon^{m-1} f(u_\varepsilon) - h(w) \le h(u_\varepsilon^m) - h(w) = (u_\varepsilon^m - w)\int_0^1 h'(\theta u_\varepsilon^m + (1-\theta)w)\mathrm{d}\theta = C_\varepsilon(x,t)\phi.$$

Then ϕ satisfies the following inequalities

$$\begin{cases} \phi_t - m(\phi + w)^{\frac{m-1}{m}}\Delta\phi - C_\varepsilon(x,t)\phi \le 0 & \text{in } \Omega \times [0, T_1], \\ \phi(x,t) \le \varepsilon^m - |(u_0(x)+1)^m|_{L^\infty}(\Omega) \le 0 & \text{in } \partial\Omega \times [0, T_1], \\ \phi(x,0) = (u_{0\varepsilon}(x) + \varepsilon)^m - |(u_0(x)+1)^m|_{L^\infty(\Omega)} \le 0 & \text{in } \bar{\Omega}. \end{cases}$$

By comparison theorem, we can have $\phi \le 0$ on $(\Omega \times (0, T_1))$. Furthermore, it follows that

$$|u^m|_{L^\infty(\Omega\times(0,T_1))} \le \max_{t\in[0,T_1]} w(t).$$

Let $T' = \frac{T_1}{2}$, $M = w(T')$, we derive that

$$|u_\varepsilon^m|_{L^\infty(\Omega\times(0,T'))} \le M.$$

Second, we may derive that

$$\int_0^{T'}\int_\Omega |\nabla u_\varepsilon^m|^2 \mathrm{d}x\mathrm{d}t \le C_1, \tag{2.9}$$

$$\int_0^{T'}\int_\Omega \left|\frac{\partial u_\varepsilon^{\frac{m+1}{2}}}{\partial t}\right|^2 \mathrm{d}x\mathrm{d}t \le C_2, \tag{2.10}$$

where C_1, and C_2 only depend on T'. Multiplying (2.3) by u_ε^m and integrating in $\Omega\times(0,T')$, we have

$$\begin{aligned}&\frac{1}{m+1}\int_\Omega u_\varepsilon^{m+1}(x,T')\mathrm{d}x-\frac{1}{m+1}\int_\Omega (u_{0\varepsilon}+\varepsilon)^{m+1}\mathrm{d}x\\ =&-\int_0^{T'}\int_\Omega |\nabla u_\varepsilon^m|^2\mathrm{d}x\mathrm{d}t+\int_0^{T'}\int_\Omega f u_\varepsilon^m \mathrm{d}x\mathrm{d}t.\end{aligned}$$

By (2.6), it follows that

$$\begin{aligned}\int_0^{T'}\int_\Omega |\nabla u_\varepsilon^m|^2\mathrm{d}x\mathrm{d}t \le& \frac{1}{m+1}\int_\Omega (u_0(x)+\varepsilon)^{m+1}\mathrm{d}x-\frac{1}{m+1}\int_\Omega u_\varepsilon^{m+1}(x,T')\mathrm{d}x\\ &+M\int_0^{T'}\int_\Omega f(x,t)\mathrm{d}x\mathrm{d}t=C_1.\end{aligned}$$

Multiplying (2.3) by $mu_\varepsilon^{m-1}u_{\varepsilon t}$ and integrating over $\Omega\times(0,T')$, we obtain that

$$\begin{aligned}\int_0^{T'}\int_\Omega m u_\varepsilon^{m-1}|u_{\varepsilon t}|^2\mathrm{d}x\mathrm{d}t =& -\frac{1}{2}\frac{\partial}{\partial t}\int_0^{T'}\int_\Omega |\nabla u_\varepsilon^m|^2\mathrm{d}x\mathrm{d}t + m\int_0^{T'}\int_\Omega f u_\varepsilon^{m-1}u_{\varepsilon t}\mathrm{d}x\mathrm{d}t\\ =& -\frac{1}{2}\int_\Omega |\nabla u_\varepsilon^m(x,T')|^2\mathrm{d}x+\frac{1}{2}\int_\Omega |\nabla u_\varepsilon^m(x,0)|^2\mathrm{d}x\\ &+\int_0^{T'}\int_\Omega \sqrt{m}\sqrt{u_\varepsilon^{m-1}}u_{\varepsilon t}\sqrt{m}\sqrt{u_\varepsilon^{m-1}}f\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By Cauchy inequality, we have

$$\int_0^{T'}\int_\Omega m u_\varepsilon^{m-1}|u_{\varepsilon t}|^2\mathrm{d}x\mathrm{d}t \le m\int_0^{T'}\int_\Omega u_\varepsilon^{m-1}|f|^2\mathrm{d}x\mathrm{d}t+|\nabla u_0^m|_{L^2(\Omega)}.$$

Furthermore, we can conclude from the above inequality that

$$\int_0^{T'}\int_\Omega \left|\frac{\partial u_\varepsilon^{\frac{m+1}{2}}}{\partial t}\right|^2\mathrm{d}x\mathrm{d}t=\frac{(m+1)^2}{4m}\int_0^{T'}\int_\Omega m u_\varepsilon^{m-1}|u_{\varepsilon t}|^2\mathrm{d}x\mathrm{d}t\le C_2.$$

Finally, inequalities (2.6), (2.9), and (2.10) imply that there is a subsequence $\{u_{\varepsilon_k}\}\subset\{u_\varepsilon\}$ and a function $u\in L^\infty(\Omega\times(0,T'))$ such that as $\varepsilon_k\to 0$

$$u_{\varepsilon_k}\to u \quad \text{a.e. on} \quad \Omega\times(0,T'),$$

$$\frac{\partial u_{\varepsilon_k}^{\frac{m+1}{2}}}{\partial t}\rightharpoonup\frac{\partial u^{\frac{m+1}{2}}}{\partial t} \quad \text{in} \quad L^2(\Omega\times(0,T')),$$

$$\nabla u_{\varepsilon_k}^m\rightharpoonup\nabla u^m \quad \text{in} \quad L^2(\Omega\times(0,T')).$$

By Definition 2.1 and equation (2.1)–(2.3), Lemma 2.1 follows by a standard limiting process.

By the idea of Vitillaro in [15], we can have the following two lemmas. □

Lemma 2.2 Let u be a solution of problem (1.1). Assume that $E(0) < E_1$ and $\|\nabla u_0^m\|_2 > \alpha_1$. Then there exists a positive constant $\alpha_2 > \alpha_1$ such that

$$\|\nabla u^m\|_2 > \alpha_2, \quad \forall t \geq 0, \tag{2.11}$$

and

$$\left(r\int_\Omega F(u)\mathrm{d}x\right)^{\frac{1}{r}} \geq B\alpha_2, \quad \forall t \geq 0. \tag{2.12}$$

Proof By (1.3) and (1.5), we have

$$E(t) \geq \frac{1}{2}\|\nabla u^m\|_2^2 - \frac{B^r}{r}\|\nabla u^m\|_2^r := \frac{1}{2}\alpha^2 - \frac{1}{r}B^r\alpha^r := g(\alpha), \tag{2.13}$$

where $\alpha = \|\nabla u^m\|_2$. It is easy to verify that the function g is increasing for $0 < \alpha < \alpha_1$; decreasing for $\alpha > \alpha_1$; $g(\alpha) \to -\infty$ as $\alpha \to +\infty$, and $g(\alpha_1) = E_1$, where α_1 is given in (1.4). Since $E(0) < E_1$, there exists an $\alpha_2 > \alpha_1$ such that $g(\alpha_2) = E(0)$. Let $\alpha_0 = \|\nabla u_0^m\|_p > \alpha_1$, then by (1.6), we have $g(\alpha_0) \leq E(0) = g(\alpha_2)$, which implies that $\alpha_0 \geq \alpha_2$. To establish (2.11), we argue by contradiction that $\|\nabla u(\cdot, t_0)\|_2 < \alpha_2$ for some $t_0 > 0$. By the continuity of $\|\nabla u^m(\cdot, t_0)\|_2$, we can choose t_0 such that $\|\nabla u^m(\cdot, t_0)\|_p > \alpha_1$. It follows from (2.13) that

$$E(t_0) \geq g(\|\nabla u^m(\cdot, t_0)\|_2) > g(\alpha_2) = E(0).$$

This is impossible since $E(t) \leq E(0)$ for all $t \geq 0$. Hence (2.11) is established.

To prove (2.12), we exploit (1.5) to see that

$$\frac{1}{2}\|\nabla u^m\|_2 \leq E(0) + \int_\Omega F(u)\mathrm{d}x. \tag{2.14}$$

Consequently,

$$\int_\Omega F(u)\mathrm{d}x \geq \frac{1}{2}\|\nabla u^m\|_2 - E(0) \geq \frac{1}{2}\alpha_2^2 - g(\alpha_2) = \frac{1}{r}B^r\alpha_2^r.$$

This completes the proof of Lemma 2.2. □

In the remainder of this section, we consider the case that $E(0) < E_1$ and $\|\nabla u_0^m\|_2 > \alpha_1$, we set

$$H(t) = E_1 - E(t), t \geq 0. \tag{2.15}$$

Then we have

Lemma 2.3 For all $t > 0$,

$$0 < H(0) \leq H(t) \leq \int_\Omega F(u)\mathrm{d}x. \tag{2.16}$$

Proof By (1.6), we can see that $H' \geq 0$. Thus

$$H(t) \geq H(0) = E_1 - E(0) > 0. \tag{2.17}$$

From (1.5), (2.15) we obtain

$$H(t) = E_1 - \frac{1}{2}\|\nabla u^m\|_2^2 + \int_\Omega F(u)\mathrm{d}x.$$

Exploiting (1.4) and (2.11), we have

$$E_1 - \frac{1}{2}\|\nabla u^m\|_2^2 \le E_1 - \frac{1}{2}\alpha_1^2 = -\frac{1}{r}B^r\alpha_1^r < 0, \quad \forall t \ge 0.$$

Hence

$$H(t) \le \int_\Omega F(u)\mathrm{d}x, \quad \forall t \ge 0. \qquad \square$$

3 Main Result and Proof

In this section, we prove the main result by the energy method.

Theorem 3.1 Assume that $N > 2$, $2 < r \le \frac{2N}{N-2}$, let $f(s)$ satisfy (1.2), (1.3), (2.2) and

$$s^m f(s) \ge rF(s) \ge | s |^{mr} . \tag{3.1}$$

Furthermore, assume that $u_0^m \ge 0$ and

$$E(0) < E_1. \tag{3.2}$$

Then the solution $u(x,t)$ of problem (1.1) blows up in finite time.

Proof Define

$$G(t) = \frac{1}{m+1}\int_\Omega u^{m+1}(x,t)\mathrm{d}x, \tag{3.3}$$

then

$$G'(t) = \int_\Omega u^m f(u)\mathrm{d}x - \int_\Omega | \nabla u^m |^2 \mathrm{d}x. \tag{3.4}$$

We replace $\int_\Omega | \nabla u^m |^2 \mathrm{d}x$ by (1.5) and (2.15), then (3.4) is equivalent to

$$\begin{aligned} G'(t) &= \int_\Omega u^m f(u)\mathrm{d}x - 2E(t) - 2\int_\Omega F(u)\mathrm{d}x \\ &= \int_\Omega u^m f(u)\mathrm{d}x - 2\int_\Omega F(u)\mathrm{d}x + 2H(t) - 2E_1. \end{aligned} \tag{3.5}$$

By using (1.4) and (2.12), we have

$$\begin{aligned} 2E_1 &= (r-2)\frac{1}{r}\alpha_1^2 = (r-2)\frac{1}{r}B^r\alpha_1^r \\ &= \frac{\alpha_1^r}{\alpha_2^r}(r-2)\frac{1}{r}B^r\alpha_2^r \le \frac{\alpha_1^r}{\alpha_2^r}(r-2)\int_\Omega F(u)\mathrm{d}x. \end{aligned} \tag{3.6}$$

It follows from (3.1) (3.5) and (3.6) that

$$\begin{aligned} G'(t) &\ge \int_\Omega u^m f(u)\mathrm{d}x - \left[\frac{\alpha_1^r}{\alpha_2^r}(r-2)+2\right]\int_\Omega F(u)\mathrm{d}x + 2H(t) \\ &\ge \int_\Omega rF(u)\mathrm{d}x - \left[\frac{\alpha_1^r}{\alpha_2^r}(r-2)+2\right]\int_\Omega F(u)\mathrm{d}x + 2H(t) \\ &= C\int_\Omega F(u)\mathrm{d}x + 2H(t) \ge 0, \end{aligned} \tag{3.7}$$

where $C = \big(1 - \frac{\alpha_1^r}{\alpha_2^r}\big)(r-2) > 0$.

Next we estimate $G^{\frac{mr}{m+1}}(t)$. By Hölder$'s$ inequality, we get

$$G^{\frac{mr}{m+1}} \le k\|u^m\|_r^r \le rk\int_\Omega F(u)\mathrm{d}x, \tag{3.8}$$

where $k = (\frac{1}{m+1})^{\frac{mr}{m+1}} \mid \Omega \mid^{\frac{mr}{m+1}-1}$. By (3.7) and (3.8), we have

$$G'(t) \ge \gamma G^{\frac{mr}{m+1}}(t), \tag{3.9}$$

where $\gamma = C/rk$. Integrating (3.9) then yields

$$G^{\frac{mr}{m+1}-1}(t) \ge \frac{1}{G^{1-\frac{mr}{m+1}}(0) - (\frac{mr}{m+1}-1)\gamma t}.$$

Therefore $G(t)$ blows up in a time $T^* \le \frac{G^{1-\frac{mr}{m+1}}(0)}{(\frac{mr}{m+1}-1)\gamma}$, so does $u(x,t)$. □

References

[1] Wu Z Q, Zhao J N, Yin J X, Li H L. Nonlinear Diffusion Equations. Singapore: World Scientific, 2001

[2] Galaktionov V A, Vazquez J L. The problem of blow-up in nonlinear parabolic equations. Discrete Contin Dynam System, 2002, **8**(2): 399–433

[3] Ferreira R, Vazquez J L. Extinction behavior for fast diffusion equations with absorption. Nonlinear Anal, 2001, **43**: 943–985

[4] Li Y X, Wu J C. Extinction for fast diffusion equations with nonlinear sources. Electron J Differ Equ, 2005, **23**: 1–7

[5] Han Y Z, Gao W J. Extinction for a fast diffusion equation with a nonlinear nonlocal source. Arch Math, 2011, **97**: 353–363

[6] Peietier L A, Junning Z. Source-type solutions of porous media equation with absorption: The fast diffusion case. Nonlinear Anal, 1991, **17**: 991–1009

[7] Leoni G. A very singular solution for the porous media equation $u_t = \Delta u^m - u^p$ when $0 < m < 1$. J Diff Equ, 1996, **132**: 353–376

[8] Mochizuki K, Suzuki R. Critical exponent and critical blow up for quasi-linear parabolic equations. Israel J Math, 1997, **98**: 141–156

[9] Mu C L, Zeng R, Chen B T. Blow-up phenomena for a doubly degenerate equation with positive initial energy. Nonlinear Anal, 2010, **72**: 782–793

[10] Liu W J, Wang M X. Blow-up of the solution for a p-Laplaican equation with positive initial energy. Acta Appl Math, 2008, **103**: 141–146

[11] Gao W J, Han Y Z. Blow-up of a nonlocal semilinear parabolic equation with positive initial energy. Appl Math Lett, 2011, **24**: 784–788

[12] Wang J, Gao W J. Existence of solutions to a class of doubly degenerate equations and blow up with vanishing intial energy. J Math Res Exposition, 2007, **27**: 161–168 (in Chinese)

[13] Ladyzenskaja O, Solonnikov V, Uraltseva N. Linear and Quasilinear Equations of Parabolic Type. Transl Math Monogr. Providence, RI: American Mathematical Society, 1968

[14] Zhong C K, Fan X L, Chen W Y. Nonlinear Functinal Analysis. Lanzhou: Lanzhou University Press, 2004 (in Chinese)

[15] Vitillaro E. Global nonexistence theorems for a class of evolution equations with dissipation. Arch Ration Mech Anal, 1999, **149**: 155–182

[16] Mu C L, Hu X G, Li Y H, Cui Z J. Blow-up and global existence for a coupled system of degenerate parabolic equations in a bounded domain. Acta Math Sci, 2007, **27B**(1): 92–106