2021,**41A**(5):1465–1491

数学物理学报 Acta Mathematica Scientia

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二维 Mindlin-Timoshenko 板系统的稳定性与最优性

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摘要：该文研究的是具有局部边界控制的二维 Mindlin-Timoshenko 板系统，运用滚动时域法，将无限时域最优性问题转化为有限时域的最优性问题进行研究. 借助乘子法技巧，首先对每一有限时域系统的解做先验估计，并得到能观性不等式，进而证明了系统能量是一致指数衰减的. 进一步，借助对偶系统，应用变分原理和 Bellman 最优性原理，得到了无限时域系统的次优性条件，并证明了最优轨线也是指数衰减的.

关键词：Mindlin-Timoshenko 板；滚动时域法；最优性；指数衰减.

MR(2010) 主题分类：35B37; 93B52 **中图分类号**：O231.4 **文献标识码**：A

文章编号：1003-3998(2021)05-1465-27

1 引言

近三十年来，对于有限维系统在滚动时域控制下的稳定性与最优性，已受到许多学者的广泛研究，并得到了许多深刻的结果. 然而对于无穷维控制系统，近十年才引起学者们的关注. 从目前的研究结果来看，对于连续时间的无穷维动态系统的最优性，还没有统一的理论结果. 但是，对于 Timoshenko 梁系统的稳定性的研究已有许多很好的结果. 就我们的知识而言，关于 Timoshenko 梁系统的最优性结果仍较少，参见文献 [1–4]; 与此同时，关于板 (或 Mindlin-Timoshenko 板) 系统的稳定性研究的结果已有丰富的结果，参见文献 [5–17]. 然而，具有滚动时域控制的系统的稳定性与最优性结果就比较少，Azmi 和 Kunisch 在文献 [18–19] 中研究了具有滚动时域控制的 Burgers 方程和波方程的稳定性与最优性. 基于此，本文研究的是 Mindlin-Timoshenko 板系统的最优性与稳定性.

设 Ω 是 $\mathbb{R}^2$ 中的有界开集，$\partial\Omega=\Gamma=\Gamma_0\cup\Gamma_1(\Gamma_0\cap\Gamma_1=\emptyset)$ 满足 Lipschitz 边界条件，且 Γ_1 是具有非空内部的闭集，$\Gamma_0\neq\emptyset$ 是相对开的.

我们考虑下面具有边界控制的无限时域的二维 Mindlin-Timosahenko 系统

$$
\begin{cases}
\rho_1\psi_{tt}-D(\psi_{xx}+\dfrac{1-\mu}{2}\psi_{yy}+\dfrac{1+\mu}{2}\phi_{xy})+K(\psi+\omega_x)=0, & (x,y,t)\in\Omega\times\mathbb{R}^+,\\
\rho_1\phi_{tt}-D(\phi_{yy}+\dfrac{1-\mu}{2}\phi_{xx}+\dfrac{1+\mu}{2}\psi_{xy})+K(\phi+\omega_y)=0, & (x,y,t)\in\Omega\times\mathbb{R}^+,\\
\rho_2\omega_{tt}-K[(\psi+\omega_x)_x+(\phi+\omega_y)_y]=0, & (x,y,t)\in\Omega\times\mathbb{R}^+,
\end{cases}
\tag{1.1}
$$

收稿日期: 2020-02-21; 修订日期: 2021-04-10

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基金项目：国家自然科学基金 (61374096)

Supported by the NSFC(61374096)

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其中，$\rho_1=\frac{\rho h^3}{2}, \rho_2=\rho h$, ρ 是密度，h 是板的厚度，$\mu\in(0,\frac{1}{2})$ 是 Poisson 比，$D=\frac{Eh^3}{12(1-\mu^2)}$ (E 杨氏模量) 表示弹性模量，$K=\frac{kEh}{2(1+\mu)}$ (k 剪切校正，E 杨氏模量) 表示剪切模量，函数 ψ,ϕ 和 ω 依赖于 $(x,y,t)\in\Omega\times\mathbb{R}^+$ 表示板的全转角和板的横向位移.

边界条件和初始条件

$$\begin{cases}\psi=\phi=\omega=0, & (x,y,t)\in\Gamma_0\times\mathbb{R}^+,\\ D[\nu_1\psi_x+\mu\nu_1\phi_y+\dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_2]=u_1, & (x,y,t)\in\Gamma_1\times\mathbb{R}^+,\\ D[\nu_2\phi_y+\mu\nu_2\psi_x+\dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_1]=u_2, & (x,y,t)\in\Gamma_1\times\mathbb{R}^+,\\ K(\dfrac{\partial\omega}{\partial\nu}+\nu_1\psi+\nu_2\phi)=u_3, & (x,y,t)\in\Gamma_1\times\mathbb{R}^+,\\ (\psi(x,y,0),\phi(x,y,0),\omega(x,y,0))=(\psi_{01},\phi_{01},\omega_{01}), & (x,y)\in\Omega,\\ (\psi_t(x,y,0),\phi_t(x,y,0),\omega_t(x,y,0))=(\psi_{02},\phi_{02},\omega_{02}), & (x,y)\in\Omega,\end{cases}\tag{1.2}$$

其中，$\nu=(\nu_1,\nu_2)$ 是 Γ_1 的单位外法向量，$U(t)=(u_1,u_2,u_3)$ 是控制变量. 与系统 (1.1)–(1.2) 相关的具体技术细节请参阅文献 [8].

定义系统 (1.1) 在 t 时刻的能量为

$$\begin{aligned}E(t)=\frac{1}{2}\int_\Omega&[D(|\psi_x|^2+|\phi_y|^2+2\mu\psi_x\phi_y+\frac{1-\mu}{2}|\psi_y+\phi_x|^2)+K(|\psi+\omega_x|^2\\&+|\phi+\omega_y|^2)]\mathrm{d}x\mathrm{d}y+\frac{1}{2}\int_\Omega[\rho_1(|\psi_t|^2+|\varphi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

为了叙述方便起见，对足够光滑的 $\Phi=(\psi,\phi,\omega),\widehat{\Phi}=(\widehat{\psi},\widehat{\phi},\widehat{\omega})$, 定义如下线性算子及双线性泛函

$$\begin{gathered}L_1\{\psi,\phi,\omega\}=D(\psi_{xx}+\frac{1-\mu}{2}\psi_{yy}+\frac{1+\mu}{2}\phi_{xy})-K(\psi+\omega_x),\\ L_2\{\psi,\phi,\omega\}=D(\phi_{yy}+\frac{1-\mu}{2}\phi_{xx}+\frac{1+\mu}{2}\psi_{xy})-K(\phi+\omega_y),\\ L_3\{\psi,\phi,\omega\}=K[(\psi+\omega_x)_x+(\phi+\omega_y)_y],\\ F_1\{\psi,\phi\}=D[\nu_1\psi_x+\mu\nu_1\phi_y+\frac{1-\mu}{2}(\psi_y+\phi_x)\nu_2],\\ F_2\{\psi,\phi\}=D[\nu_2\phi_y+\mu\nu_2\psi_x+\frac{1-\mu}{2}(\psi_y+\phi_x)\nu_1],\\ F_3\{\psi,\phi,\omega\}=K(\frac{\partial\omega}{\partial\nu}+\nu_1\psi+\nu_2\phi),\end{gathered}$$

$$\begin{aligned}a(\psi,\phi,\omega;\widehat{\psi},\widehat{\phi},\widehat{\omega})=D\int_\Omega&[\psi_x\widehat{\psi}_x+\phi_y\widehat{\phi}_y+\mu(\widehat{\psi}_x\phi_y+\psi_x\widehat{\phi}_y)+\frac{1-\mu}{2}(\psi_y+\psi_x)(\widehat{\psi}_y+\widehat{\psi}_x)]\mathrm{d}x\mathrm{d}y\\&+K\int_\Omega[(\psi+\omega_x)(\widehat{\psi}+\widehat{\omega}_x)+(\phi+\omega_y)(\widehat{\phi}+\widehat{\omega}_y)]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

引入函数空间 $W=\{(\psi,\phi,\omega)\in[H^1(\Omega)]^3|\psi=\phi=\omega=0,(x,y)\in\Gamma_0\}$, 赋予范数 $\|(\psi,\phi,\omega)\|_W^2=a(\psi,\phi,\omega;\psi,\phi,\omega)$, 其中 $H^1(\Omega)$ 是一阶 Sobolev 空间 [20].

空间 $H=(L^2_{\rho_1}(\Omega))^2\times L^2_{\rho_2}(\Omega)$, 赋予范数 $\|(p,q,r)\|_H^2=\int_\Omega[\rho_1(|p|^2+|q|^2)+\rho_2|r|^2]\mathrm{d}x\mathrm{d}y$, 那么 W,H 都是 (复)Hilbert 空间.

定义空间 $\mathcal{H}=W\times H$, 赋予范数 $\|(\psi,\phi,\omega;p,q,r)\|_{\mathcal{H}}^2=\|(\psi,\phi,\omega)\|_W^2+\|(p,q,r)\|_H^2$, 因此 $\mathcal{H}$ 也是一个 Hilbert 空间.

定义 $\mathcal{U}=(L^2(\Gamma_1))^3$, 赋予范数 $\|U\|_{\mathcal{U}}^2=\int_{\Gamma_1}(|u_1|^2+|u_2|^2+|u_3|^2)\mathrm{d}\Gamma_1$, 那么 $\mathcal{U}$ 也是一个 Hilbert 空间.

在状态空间 $\mathcal{H}$, 控制空间 $\mathcal{U}$ 中，记 $\Phi=(\psi,\phi,\omega),\mathcal{Y}=(\Phi,\Phi_t),\mathcal{Y}_0=(\Phi_{01},\Phi_{02}),\mathcal{Y},\mathcal{Y}_0\in\mathcal{H}$ 和 $U(t)=(u_1,u_2,u_3)\in L^2((0,\infty);\mathcal{U})$. 考虑具有下面的性能指标 (代价函数)

$$J_\infty(\mathcal{Y}_0,U)=\int_0^\infty \ell(\mathcal{Y}(t),U(t))\mathrm{d}t, \tag{1.3}$$

$$\ell(\mathcal{Y}(t),U(t))=\frac{1}{2}\|\mathcal{Y}(t)\|_{\mathcal{H}}^2+\frac{\beta}{2}\|U(t)\|_{\mathcal{U}}^2, \tag{1.4}$$

这里， β 是一个正常数.

本文采用文献 [17] 的滚动时域控制方法，即对任意给定采样时间 $\delta>0$ 和一个合适的预测时间 $T>\delta$, 定义采样时间序列 $t_k=k\delta,k=0,1,\cdots$, 对于每一个时刻 t_k, 在有限预测区间 $[t_k,t_k+T]$ 内，求解开环最优控制问题. 由此我们将无限时域的最优控制问题转化为有限时间 $[t_k,t_k+T]$ 内最优控制问题. $\mathcal{Y}(t),U(t)$ 分别表示滚动时域的状态和控制变量, $\mathcal{Y}_T^*(\cdot,\mathcal{Y}_0,t_0),U_T^*(\cdot,\mathcal{Y}_0,t_0)$ 分别表示在有限时域 $[0,T]$ 内的最优控制问题的最优状态和最优控制. 不难看出，通过不断改变预测区间，我们将获得新的最优控制以及最优状态轨线. 具体做法如下：

给定采样时间 $\delta>0$ 以及预测时间 $T>\delta$, 当 $k=0$ 时， $t_0=0$, $\mathcal{Y}(t_0)=\mathcal{Y}_0$, 在时间 $[t_k,t_k,+T]$ 内求解以下开环最优问题

$$\min_{U\in L^2((t_k,t_k+T);\mathcal{U})} J_T(\mathcal{Y}(t_k),U)=\min_{U\in L^2((t_k,t_k+T);\mathcal{U})}\int_{t_k}^{t_k+T}\ell(\mathcal{Y}(t),U(t))\mathrm{d}t, \tag{1.5}$$

满足

$$\begin{cases}
\rho_1\psi_{tt}-D(\psi_{xx}+\dfrac{1-\mu}{2}\psi_{yy}+\dfrac{1+\mu}{2}\phi_{xy})+K(\psi+\omega_x)=0,(x,y,t)\in\Omega\times(t_k,t_k+T),\\
\rho_1\phi_{tt}-D(\phi_{yy}+\dfrac{1-\mu}{2}\phi_{xx}+\dfrac{1+\mu}{2}\psi_{xy})+K(\phi+\omega_y)=0,(x,y,t)\in\Omega\times(t_k,t_k+T),\\
\rho_2\omega_{tt}-K[(\psi+\omega_x)_x+(\phi+\omega_y)_y]=0,\quad (x,y,t)\in\Omega\times(t_k,t_k+T),\\
\psi=\phi=\omega=0,\quad (x,y,t)\in\Gamma_0\times(t_k,t_k+T),\\
D[\nu_1\psi_x+\mu\nu_1\phi_y+\dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_2]=u_1,\quad (x,y,t)\in\Gamma_1\times(t_k,t_k+T),\\
D[\nu_2\phi_y+\mu\nu_2\psi_x+\dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_1]=u_2,\quad (x,y,t)\in\Gamma_1\times(t_k,t_k+T),\\
K(\dfrac{\partial\omega}{\partial\nu}+\nu_1\psi+\nu_2\phi)=u_3,\quad (x,y,t)\in\Gamma_1\times(t_k,t_k+T),\\
(\psi(x,y,t_k),\phi(x,y,t_k),\omega(x,y,t_k))=(\psi_{t_k1},\phi_{t_k1},\omega_{t_k1}),\quad (x,y)\in\Omega,\\
(\psi_t(x,y,t_k),\phi_t(x,y,t_k),\omega_t(x,y,t_k))=(\psi_{t_k2},\phi_{t_k2},\omega_{t_k2}),\quad (x,y)\in\Omega.
\end{cases} \tag{1.6}$$

本文将无限时域最优性问题转化为有限时域的最优性问题. 借助乘子技巧，对每一有限时域系统的解做先验估计，证明了系统能量是一致指数衰减的. 进一步，借助对偶方法、变分原理和 Bellman 最优性原理，得到系统的次优性条件，由此证明了最优轨线是指数衰减的.

2 弱解的先验估计

首先，我们考虑任意初始值 $\mathcal{Y}_0 \in \mathcal{H}$ 在有限时域内的最优控制问题 (OCP)

$$\min_{U\in L^2((0,T);\mathcal{U})} J_T(\mathcal{Y}_0, U) = \min_{U\in L^2((0,T);\mathcal{U})} \int_0^T \ell(\mathcal{Y}(t), U(t))\mathrm{d}t, \tag{2.1}$$

满足

$$\begin{cases} \rho_1\psi_{tt} - D(\psi_{xx} + \dfrac{1-\mu}{2}\psi_{yy} + \dfrac{1+\mu}{2}\phi_{xy}) + K(\psi+\omega_x) = 0, (x,y,t)\in\Omega\times(0,T),\\ \rho_1\phi_{tt} - D(\phi_{yy} + \dfrac{1-\mu}{2}\phi_{xx} + \dfrac{1+\mu}{2}\psi_{xy}) + K(\phi+\omega_y) = 0, (x,y,t)\in\Omega\times(0,T),\\ \rho_2\omega_{tt} - K[(\psi+\omega_x)_x + (\phi+\omega_y)_y] = 0, \quad (x,y,t)\in\Omega\times(0,T),\\ \psi=\phi=\omega=0, \quad (x,y,t)\in\Gamma_0\times(0,T),\\ D[\nu_1\psi_x + \mu\nu_1\phi_y + \dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_2] = u_1, \quad (x,y,t)\in\Gamma_1\times(0,T),\\ D[\nu_2\phi_y + \mu\nu_2\psi_x + \dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_1] = u_2, \quad (x,y,t)\in\Gamma_1\times(0,T),\\ K(\dfrac{\partial\omega}{\partial\nu} + \nu_1\psi + \nu_2\phi) = u_3, \quad (x,y,t)\in\Gamma_1\times(0,T),\\ (\psi(x,y,0),\phi(x,y,0),\omega(x,y,0)) = (\psi_{01},\phi_{01},\omega_{01}), \quad (x,y)\in\Omega,\\ (\psi_t(x,y,0),\phi_t(x,y,0),\omega_t(x,y,0)) = (\psi_{02},\phi_{02},\omega_{02}), \quad (x,y)\in\Omega. \end{cases} \tag{2.2}$$

定义 2.1 (值函数) 对每一初始状态 $\mathcal{Y}_0 \in \mathcal{H}$, 定义无限时域值函数： $V_\infty : \mathcal{H} \to \mathbb{R}^+$,

$$V_\infty(\mathcal{Y}_0) = \inf_{U\in L^2((0,+\infty);\mathcal{U})} \{J_\infty(\mathcal{Y}_0, U)\}.$$

其中 $(\mathcal{Y}, U)$ 满足系统 (1.1) 和 (1.2)；类似地，定义有限时域的值函数： $V_T : \mathcal{H} \to \mathbb{R}^+$,

$$V_T(\mathcal{Y}_0) = \min_{U\in L^2((0,T);\mathcal{U})} \{J_T(\mathcal{Y}_0, U)\},$$

且 $(\mathcal{Y}, U)$ 满足系统 (2.2).

定义 2.2 (弱解) 给定 $T > 0, \mathcal{Y}_0 \in \mathcal{H}$, $U \in \mathcal{U}$, 那么我们称 $\Phi = (\psi,\phi,\omega)$ 为系统 (2.2) 的弱解，如果对于任意 $\widehat{\Phi} = (\widehat{\psi},\widehat{\phi},\widehat{\omega}) \in C^1([0,T]\times\Omega)$, 在 Γ_0 上， $\widehat{\Phi} = 0$, 以及对于 $\forall t\in[0,T], \tau\in[0,t]$ 使得

$$\begin{aligned} &\int_\Omega \{\rho_1[\psi_t(t,x,y)\widehat{\psi}(t,x,y) + \phi_t(t,x,y)\widehat{\phi}(t,x,y)] + \rho_2\omega_t(t,x,y)\widehat{\omega}(t,x,y)\}\mathrm{d}x\mathrm{d}y\\ &-\int_\Omega \{\rho_1[\psi_{02}\widehat{\psi}(0,x,y) + \phi_{02}\widehat{\phi}(0,x,y)] + \rho_2\omega_{02}\widehat{\omega}(0,x,y)\}\mathrm{d}x\mathrm{d}y\\ &-\int_0^t\int_\Omega \{\rho_1[\psi_t(\tau,x,y)\widehat{\psi}_t(\tau,x,y) + \phi_t(\tau,x,y)\widehat{\phi}_t(\tau,x,y)] + \rho_2\omega_t(\tau,x,y)\widehat{\omega}_t(\tau,x,y)\}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\ &+\int_0^t a(\psi,\phi,\omega;\widehat{\psi},\widehat{\phi},\widehat{\omega})\mathrm{d}\tau - \int_0^t\int_{\Gamma_1}(\widehat{\psi}u_1 + \widehat{\phi}u_2 + \widehat{\omega}u_3)\mathrm{d}\Gamma\mathrm{d}\tau = 0. \end{aligned} \tag{2.3}$$

命题 2.1 给定 $T>0,\mathcal{Y}_0\in\mathcal{H}$, $U\in L^2([0,T];\mathcal{U})$, 则系统 (2.2) 存在唯一的弱解 $\Phi=(\psi,\phi,\omega)\in C^0([0,T];W)\cap C^1([0,T],H)$ 满足

$$\begin{aligned}&\|\Phi\|_{C^0([0,T];W)}+\|\Phi_t\|_{C^0([0,T];H)}+\|\Phi_{tt}\|_{L^2([0,T];W^*)}\\ \le\ & C(\|\Phi_{01}\|_W+\|\Phi_{02}\|_H+\|U\|_{L^2([0,T];\mathcal{U})}),\end{aligned}\tag{2.4}$$

其中 C(常数 C 在不同地方表示不一样的常数) 是与 Φ_{01},Φ_{02},U 无关的正常数.

证 记 $I(t)=a(\psi,\phi,\omega;\psi,\phi,\omega)+\int_\Omega[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y$, 则有

$$\begin{aligned}\frac{\mathrm{d}I(t)}{\mathrm{d}t}&=\frac{\partial a(\psi,\phi,\omega;\psi,\phi,\omega)}{\partial t}+2\int_\Omega[\rho_1(\psi_t\psi_{tt}+\phi_t\phi_{tt})+\rho_2\omega_t\omega_{tt}]\mathrm{d}x\mathrm{d}y\\&=2a(\psi,\phi,\omega;\psi_t,\phi_t,\omega_t)+2\int_\Omega[\psi_tL_1\{\psi,\phi,\omega\}+\phi_tL_2\{\psi,\phi,\omega\}+\omega_tL_3\{\psi,\phi,\omega\}]\mathrm{d}x\mathrm{d}y\\&=2\int_{\Gamma_1}[\psi_tF_1\{\psi,\phi\}+\phi_tF_2\{\psi,\phi\}+\omega_tF_3\{\psi,\phi,\omega\}]\mathrm{d}\Gamma_1\\&\le2\varepsilon\int_{\Gamma_1}[|\psi_t|^2+|\phi_t|^2+|\omega_t|^2]\mathrm{d}\Gamma_1+\frac{1}{2\varepsilon}\int_{\Gamma_1}[|u_1|^2+|u_2|^2+|u_3|^2]\mathrm{d}\Gamma_1.\end{aligned}\tag{2.5}$$

设 $f=(f_1,f_2):\overline{\Omega}\to\mathbb{R}^2$ 是 C^1 的向量值函数，且在 $\overline{\Gamma}_0$ 上有 $f=0$, 则存在 $\sigma>0$, 在 Γ_1 上几乎处处满足 $f\cdot\nu\ge\sigma$.

记 $2M_0=\min\{\rho_1\rho_1\},R_0=\max\limits_{(x,y)\in\overline{\Omega}}\{|f|,|f_{ix}|,|f_{iy}|\},i=1,2$, 其中， $|f|^2=f_1^2+f_2^2$. 对 $\forall t\in[0,T],\tau\in[0,t]$, 将 $f\cdot\nabla\psi,f\cdot\nabla\phi,f\cdot\nabla\omega$ 分别乘以系统 (2.2) 的前三式，在 $[0,t]\times\Omega$ 上积分并相加得

$$\begin{aligned}&\int_0^t\int_\Omega[\rho_1(\psi_{tt}f\cdot\nabla\psi+\phi_{tt}f\cdot\nabla\phi)+\rho_2\omega_{tt}f\cdot\nabla\omega]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\&-\int_0^t\int_\Omega[L_1\{\psi,\phi,\omega\}f\cdot\nabla\psi+L_2\{\psi,\phi,\omega\}f\cdot\nabla\phi+L_3\{\psi,\phi,\omega\}f\cdot\nabla\omega]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau=0.\end{aligned}\tag{2.6}$$

分部积分得

$$\begin{aligned}&\int_\Omega[\rho_1(\psi_tf\cdot\nabla\psi+\phi_tf\cdot\nabla\phi)+\rho_2\omega_tf\cdot\nabla\omega]_0^t\mathrm{d}x\mathrm{d}y\\&-\frac12\int_0^t\int_\Omega[\rho_1(f\cdot\nabla|\psi_t|^2+f\cdot\nabla|\phi_t|^2)+\rho_2f\cdot\nabla|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\&+\int_0^ta(\psi,\phi,\omega;f\cdot\nabla\psi,f\cdot\nabla\phi,f\cdot\nabla\omega)\mathrm{d}\tau\\&-\int_0^t\int_{\Gamma_1}[F_1\{\psi,\phi\}f\cdot\nabla\psi+F_2\{\psi,\phi\}f\cdot\nabla\phi+F_3\{\psi,\phi,\omega\}f\cdot\nabla\omega]\mathrm{d}\Gamma_1\mathrm{d}\tau=0,\end{aligned}$$

即

$$\begin{aligned}&\int_\Omega[\rho_1(\psi_tf\cdot\nabla\psi+\phi_tf\cdot\nabla\phi)+\rho_2\omega_tf\cdot\nabla\omega]_0^t\mathrm{d}x\mathrm{d}y\\&-\frac12\int_0^t\int_{\Gamma_1}(f\cdot\nu)[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\&+\frac12\int_0^t\int_\Omega\mathrm{div}f[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\end{aligned}$$

$$+\int_0^t a(\psi,\phi,\omega;f\cdot\nabla\psi,f\cdot\nabla\phi,f\cdot\nabla\omega)\mathrm{d}\tau$$
$$-\int_0^t\int_{\Gamma_1}[F_1\{\psi,\phi\}f\cdot\nabla\psi+F_2\{\psi,\phi\}f\cdot\nabla\phi+F_3\{\psi,\phi,\omega\}f\cdot\nabla\omega]\mathrm{d}\Gamma_1\mathrm{d}\tau=0. \tag{2.7}$$

又由于

$$\begin{aligned}
&\int_0^t a(\psi,\phi,\omega;f\cdot\nabla\psi,f\cdot\nabla\phi,f\cdot\nabla\omega)\mathrm{d}\tau\\
=&\,D\int_0^t\int_\Omega[\psi_x(f_{1x}\psi_x+f_{2x}\psi_y)+\phi_y(f_{1y}\phi_x+f_{2y}\phi_y)+\mu\psi_x(f_{1y}\phi_x+f_{2y}\phi_y)\\
&+\mu\phi_y(f_{1x}\psi_x+f_{2x}\psi_y)+\frac{1-\mu}{2}(\psi_y+\phi_x)(f_{1y}\psi_x+f_{2y}\psi_y+f_{1x}\phi_x+f_{2x}\phi_y)]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&+K\int_0^t\int_\Omega[(\psi+\omega_x)(f_{1x}\omega_x+f_{2x}\omega_y)+(\phi+\omega_y)(f_{1y}\omega_x+f_{2y}\omega_y)]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&+\frac{D}{2}\int_0^t\int_\Omega[f\cdot\nabla|\psi_x|^2+f\cdot\nabla|\phi_y|^2+\mu f\cdot\nabla(\psi_x\phi_y)+\frac{1-\mu}{2}f\cdot\nabla|\psi_y+\phi_x|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&+\frac{K}{2}\int_0^t\int_\Omega[f\cdot\nabla|\psi+\omega_x|^2+f\cdot\nabla|\phi+\omega_y|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
=&\,D\int_0^t\int_\Omega\nabla\psi\cdot\begin{pmatrix}f_{1x}&f_{1y}\\f_{2x}&f_{2y}\end{pmatrix}\begin{pmatrix}\psi_x+\mu\phi_y\\ \frac{1-\mu}{2}(\psi_x+\phi_x)\end{pmatrix}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&+D\int_0^t\int_\Omega\nabla\phi\cdot\begin{pmatrix}f_{1x}&f_{1y}\\f_{2x}&f_{2y}\end{pmatrix}\begin{pmatrix}\frac{1-\mu}{2}(\psi_x+\phi_x)\\ \phi_y+\mu\psi_x\end{pmatrix}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&+K\int_0^t\int_\Omega\nabla\omega\cdot\begin{pmatrix}f_{1x}&f_{1y}\\f_{2x}&f_{2y}\end{pmatrix}\begin{pmatrix}\psi+\omega_x)\\ \phi+\omega_y\end{pmatrix}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&+\frac{D}{2}\int_0^t\int_{\Gamma_1}(f\cdot\nu)[|\psi_x|^2+|\phi_y|^2+2\mu\psi_x\phi_y+\frac{1-\mu}{2}|\psi_y+\phi_x|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\
&+\frac{K}{2}\int_0^t\int_\Omega(f\cdot\nu)[|\psi+\omega_x|^2+|\phi+\omega_y|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&-\frac{D}{2}\int_0^t\int_\Omega\mathrm{div}f[|\psi_x|^2+|\psi_y|^2+2\mu\psi_x\phi_y+\frac{1-\mu}{2}|\psi_y+\phi_x|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&-\frac{K}{2}\int_0^t\int_\Omega\mathrm{div}f[|\psi+\omega_x|^2+|\phi+\omega_y|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau.
\end{aligned}$$

因此

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_{\Gamma_1}(f\cdot\nu)[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\
=&\int_\Omega[\rho_1(\psi_t f\cdot\nabla\psi+\phi_t f\cdot\nabla\phi)+\rho_2\omega_t f\cdot\nabla\omega]_0^t\mathrm{d}x\mathrm{d}y\\
&+\frac{1}{2}\int_0^t\int_\Omega\mathrm{div}f[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&+\int_0^t a(\psi,\phi,\omega;f\cdot\nabla\psi,f\cdot\nabla\phi,f\cdot\nabla\omega)\mathrm{d}\tau
\end{aligned}$$

$$-\int_0^t\int_{\Gamma_1}[F_1\{\psi,\phi\}f\cdot\nabla\psi+F_2\{\psi,\phi\}f\cdot\nabla\phi+F_3\{\psi,\phi,\omega\}f\cdot\nabla\omega]\mathrm{d}\Gamma_1\mathrm{d}\tau.$$

为了方便起见，记

$$\begin{aligned}
I_1 &= \int_\Omega[\rho_1(\psi_t f\cdot\nabla\psi+\phi_t f\cdot\nabla\phi)+\rho_2\omega_t f\cdot\nabla\omega]_0^t\mathrm{d}x\mathrm{d}y,\\
I_2 &= \frac{1}{2}\int_0^t\int_\Omega \mathrm{div} f[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&\quad +D\int_0^t\int_\Omega\nabla\psi\cdot\begin{pmatrix} f_{1x} & f_{1y}\\ f_{2x} & f_{2y}\end{pmatrix}\begin{pmatrix}\psi_x+\mu\phi_y\\ \dfrac{1-\mu}{2}(\psi_x+\phi_x)\end{pmatrix}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&\quad +D\int_0^t\int_\Omega\nabla\phi\cdot\begin{pmatrix} f_{1x} & f_{1y}\\ f_{2x} & f_{2y}\end{pmatrix}\begin{pmatrix}\dfrac{1-\mu}{2}(\psi_x+\phi_x)\\ \phi_y+\mu\psi_x\end{pmatrix}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&\quad +K\int_0^t\int_\Omega\nabla\omega\cdot\begin{pmatrix} f_{1x} & f_{1y}\\ f_{2x} & f_{2y}\end{pmatrix}\begin{pmatrix}\psi+\omega_x)\\ \phi+\omega_y\end{pmatrix}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&\quad -\frac{D}{2}\int_0^t\int_\Omega \mathrm{div} f[|\psi_x|^2+|\psi_y|^2+2\mu\psi_x\phi_y+\frac{1-\mu}{2}|\psi_y+\phi_x|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&\quad -\frac{K}{2}\int_0^t\int_\Omega \mathrm{div} f[|\psi+\omega_x|^2+|\phi+\omega_y|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau,\\
I_3 &= \frac{D}{2}\int_0^t\int_{\Gamma_1}(f\cdot\nu)[|\psi_x|^2+|\phi_y|^2+2\mu\psi_x\phi_y+\frac{1-\mu}{2}|\psi_y+\phi_x|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\
&\quad +\frac{K}{2}\int_0^t\int_{\Gamma_1}(f\cdot\nu)[|\psi+\omega_x|^2+|\phi+\omega_y|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\
&\quad -\int_0^t\int_{\Gamma_1}[F_1\{\psi,\phi\}f\cdot\nabla\psi+F_2\{\psi,\phi\}f\cdot\nabla\phi+F_3\{\psi,\phi,\omega\}f\cdot\nabla\omega]\mathrm{d}\Gamma_1\mathrm{d}\tau.
\end{aligned}$$

下面对 I_1,I_2,I_3 做估计.

由 Poincaré 和 Cauchy-Schwarz 不等式知，存在与 ψ,ϕ,ω 无关的正常数 C_1,C_2 使得

$$\begin{aligned}
|I_1| &\le C_1\left|\left[a(\psi,\phi,\omega;\psi,\phi,\omega)+\int_\Omega[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\right]_0^t\right|\\
&\le C_1(I(t)+I(0)),\\
|I_2| &\le \frac{R_0}{2}\int_0^t\int_\Omega[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau+\frac{3R_0}{2}\int_0^t a(\psi,\phi,\omega;\psi,\phi,\omega)\mathrm{d}\tau\\
&\le 2R_0\left[\int_0^t\int_\Omega[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau+\int_0^t a(\psi,\phi,\omega;\psi,\phi,\omega)\mathrm{d}\tau\right]\\
&= 2R_0\int_0^t I(\tau)\mathrm{d}\tau,\\
|I_3| &\le \frac{R_0C_2D^2}{2}\int_0^t\int_{\Gamma_1}[|\psi_x+\mu\phi_y|^2+(\frac{1-\mu}{2})^2|\psi_y+\phi_x|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\
&\quad +\frac{R_0C_2K^2}{2}\int_0^t\int_{\Gamma_1}[|\psi+\omega_x|^2+|\phi+\omega_y|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+C_2R_0\int_0^t\int_{\Gamma_1}[|F_1\{\psi,\phi\}|^2+|F_2\{\psi,\phi\}|^2+|F_3\{\psi,\phi,\omega\}|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\
=&\frac{C_2R_0}{2}\int_0^t\int_{\Gamma_1}[|F_1\{\psi,\phi\}|^2+|F_2\{\psi,\phi\}|^2+|F_3\{\psi,\phi,\omega\}|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\
&+C_2R_0\int_0^t\int_{\Gamma_1}[|F_1\{\psi,\phi\}|^2+|F_2\{\psi,\phi\}|^2+|F_3\{\psi,\phi,\omega\}|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\\
=&\frac{3C_2R_0}{2}\int_0^t\int_{\Gamma_1}[|u_1|^2+|u_2|^2+|u_3|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau.
\end{aligned}
$$

因此，我们有

$$
\begin{aligned}
&\int_0^t\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}\tau\\
\le&\frac{1}{\sigma M_0}\bigg[C_1(I(t)+I(0))+2R_0\int_0^tI(\tau)\mathrm{d}\tau+\frac{3C_2R_0}{2}\int_0^t\int_{\Gamma_1}[|u_1|^2+|u_2|^2+|u_3|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\bigg]\\
\le&C_3\bigg[I(t)+I(0)+\int_0^tI(\tau)\mathrm{d}\tau+\int_0^t\int_{\Gamma_1}[|u_1|^2+|u_2|^2+|u_3|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\bigg],
\end{aligned}\tag{2.8}
$$

这里， C_3 是某个正常数.

由 (2.5) 和 (2.8) 式即得

$$
I(t)\le C_4\bigg[I(0)+\int_0^tI(\tau)\mathrm{d}\tau+\int_0^t\int_{\Gamma_1}[|u_1|^2+|u_2|^2+|u_3|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\bigg],\tag{2.9}
$$

这里， C_4 是某个正常数.

由 Gronwall 不等式得

$$
I(t)\le C_5\bigg[I(0)+\int_0^t\int_{\Gamma_1}[|u_1|^2+|u_2|^2+|u_3|^2]\mathrm{d}\Gamma_1\mathrm{d}\tau\bigg],\tag{2.10}
$$

这里， C_5 是某个正常数．即

$$
\|\Phi\|_{C^0([0,T];W)}+\|\Phi_t\|_{C^0([0,T];H)}\le C_6(\|\Phi_{01}\|_W+\|\Phi_{02}\|_H+\|U\|_{L^2([0,T];\mathcal{U})}),
$$

这里， C_6 是某个正常数.

对于任意 $Z=(z_1,z_2,z_3)\in W$, 且 $\|Z\|_W=1$, 将 $Z=(z_1,z_2,z_3)$ 与系统 (2.2) 前三式在 H 中作内积，并分部积分得

$$
\begin{aligned}
&|\langle(\psi_{tt},\phi_{tt},\omega_{tt}),(z_1,z_2,z_3)\rangle_{W^*\times W}|=|((\psi_{tt},\phi_{tt},\omega_{tt}),(z_1,z_2,z_3))_H|\\
=&\left|a(\psi,\phi,\omega;z_1,z_2,z_3)-\int_{\Gamma_1}(u_1z_1+u_2z_2+u_3z_3)\mathrm{d}\Gamma_1\right|\\
\le&C_7(\|(\psi,\phi,\omega)\|_W+\|(u_1,u_2,u_3)\|_{\mathcal{U}})\|(z_1,z_2,z_3)\|_W,
\end{aligned}
$$

这里 $\langle\cdot,\cdot\rangle_{W^*\times W}$ 是对偶积， $(\cdot,\cdot)_H$ 是积， C_7 是某个正常数．因此，由 (2.10) 式得

$$
\begin{aligned}
\int_0^T\|(\psi_{tt},\phi_{tt},\omega_{tt})\|_{W^*}\mathrm{d}t&\le C\int_0^T(\|(\psi,\phi,\omega)\|_W+\|(u_1,u_2,u_3)\|_{\mathcal{U}})\mathrm{d}t\\
&\le C(\|\Phi_{01}\|_W+\|\Phi_{02}\|_H+\|U\|_{L^2([0,T];\mathcal{U})}).
\end{aligned}
$$

于是有

$$\|\Phi\|_{C^0([0,T];W)}+\|\Phi_t\|_{C^0([0,T];H)}+\|\Phi_{tt}\|_{L^2([0,T];W^*)}\le C(\|\Phi_{01}\|_W+\|\Phi_{02}\|_H+\|U\|_{L^2([0,T];\mathcal{U})}).$$

并且有

$$\int_0^t\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}\tau\le C(I(0)+\int_0^t\int_{\Gamma_1}(|u_1|^2+|u_2|^2+|u3|^2)\mathrm{d}\Gamma_1\mathrm{d}\tau. \tag{2.11}$$

命题 2.1 证毕. ▌

3 最优控制的存在唯一性

定理 3.1 对于任意 $T>0$ 以及 $\mathcal{Y}_0\in\mathcal{H}$, 最优控制问题 (OCP) 存在唯一解.

证 我们利用标准的变分原理进行证明．由目标函数 $J_T(\mathcal{Y}_0,U)$ 的定义知， $J_T(\mathcal{Y}_0,U)$ 是强制的且有下界，故必有下确界．记

$$\inf_{U\in L^2(0,T;\mathcal{U})}J_T(\mathcal{Y}_0,U)=\inf_{U\in L^2(0,T;\mathcal{U})}\int_0^T\ell(\mathcal{Y}(t),U(t))\mathrm{d}t=\theta<\infty. \tag{3.1}$$

因此，存在有界极小化序列 $\{U_n\}\subset L^2([0,T];\mathcal{U})$, 使得

$$J_T(\mathcal{Y}_0,U_n)\longrightarrow\inf_{U\in L^2(0,T;\mathcal{U})}J_T(\mathcal{Y}_0,U)=\theta. \tag{3.2}$$

进一步可知， U_n 存在弱收敛的子列，不妨仍记为 U_n ，即

$$U_n\xrightarrow{w}U^*=(u_1^*,u_2^*,u_3^*)\in L^2(0,T;\mathcal{U}). \tag{3.3}$$

由 (2.4) 式知, 对应于 $\{U_n\}$, 以及有界序列 $\{\Phi_n\}\subset L^2(0,T;W)$, $\{\Phi_{nt}\}\subset L^2(0,T;H)$, $\{\Phi_{ntt}\}\subset L^2(0,T:W^*)$, 存在子列，仍记为 $\{\Phi_n\}$, 使得

$$\begin{aligned}&\Phi_n\xrightarrow{w^*}\Phi^*,\qquad \text{in}\quad L^\infty(0,T;W),\\&\Phi_{nt}\xrightarrow{w^*}\Phi_t^*,\qquad \text{in}\quad L^\infty(0,T;H),\\&\Phi_{ntt}\xrightarrow{w^*}\Phi_{tt}^*,\qquad \text{in}\quad L^2(0,T;W^*).\end{aligned} \tag{3.4}$$

下面，我们将证明：当 $U=U^*$ 时， Φ^* 是系统 (2.2) 的弱解.

对于任意 $t\in[0,T],\tau\in[0,t]$, 以及 $\forall\widehat{\Phi}=(\widehat{\psi},\widehat{\phi},\widehat{\omega})\in C^1([0,T]\times\Omega)$, 且在 Γ_0 上，有 $\widehat{\Phi}=0$, 于是，由 (3.3) 与 (3.4) 式有

$$\begin{aligned}&\int_0^t[a(\psi_n,\phi_n,\omega_n;\widehat{\psi},\widehat{\phi},\widehat{\omega})-a(\psi^*,\phi^*,\omega^*;\widehat{\psi},\widehat{\phi},\widehat{\omega})]\mathrm{d}\tau\longrightarrow0,\\&\int_0^t\int_\Omega\rho_1\{[\psi_{nt}(x,y,\tau)-\psi_t^*(x,y,\tau)]\widehat{\psi}(x,y,\tau)+[\phi_{nt}(x,y,\tau)-\phi_t^*(x,y,\tau)]\widehat{\phi}(x,y,\tau)\}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\&+\int_0^t\int_\Omega\rho_2[\omega_{nt}(x,y,\tau)-\omega_t^*(x,y,\tau)]\widehat{\omega}(x,y,\tau)\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\longrightarrow0,\\&\int_0^t\int_\Omega\rho_1\{[\psi_{nt}(x,y,\tau)-\psi_t^*(x,y,\tau)]\widehat{\psi}_t(x,y,\tau)+[\phi_{nt}(x,y,\tau)-\phi_t^*(x,y,\tau)]\widehat{\phi}_t(x,y,\tau)\}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\&+\int_0^t\int_\Omega\rho_2[\omega_{nt}(x,y,\tau)-\omega_t^*(x,y,\tau)]\widehat{\omega}_t(x,y,\tau)\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\longrightarrow0,\\&\int_0^t\int_{\Gamma_1}[(u_{1n}-u_1^*)\widehat{\psi}+(u_{2n}-u_2^*)\widehat{\phi}+(u_{3n}-u_3^*)\widehat{\omega}]\mathrm{d}\Gamma_1\mathrm{d}\tau\longrightarrow0.\end{aligned} \tag{3.5}$$

又由 (2.4) 式得，对于任意 $t\in[0,T]$, $\{\Phi_{nt}\}$ 是 H 中的有界序列，因此，必存在弱收敛的子列，仍记为 $\{\Phi_{nt}\}$, 使得: $\Phi_{nt}\xrightarrow{w}\Phi_t\in H$. 对于任意 $t\in[0,T]$, 定义如下算子 $\Lambda: H^1(0,T;W^*)\to W^*$, 其中, W^* 为 W 的对偶空间. 那么算子 Λ 是连续的，且对于任意 $X\in W$, 有

$$\begin{aligned}(\Phi_t,X)_H&=\lim_{n\to\infty}\langle\Lambda\Phi_{nt},X\rangle_{W^*\times W}\\&=\lim_{n\to\infty}\langle\Phi_{nt},\Lambda^*X\rangle_{H^1(0,T;W^*)\times(H^1(0,T;W^*))^*}\\&=\langle\Phi_t^*,\Lambda^*X\rangle_{H^1(0,T;W^*)\times(H^1(0,T;W^*))^*}\\&=\langle\Lambda\Phi_t^*,X\rangle_{W^*\times W}.\end{aligned}\tag{3.6}$$

这里, Λ^* 表示 Λ 的伴随算子，且 $\Lambda^*: W^*\longrightarrow(H^1(0,T;W^*))^*$. 因此，对于任意 $t\in[0,T]$, 以及 $\forall\widehat{\Phi}=(\widehat{\psi},\widehat{\phi},\widehat{\omega})\in C^1([0,T]\times\Omega)$, 且在 Γ_0 上有 $\widehat{\Phi}=0$, 我们有

$$\begin{aligned}&\int_\Omega\{\rho_1[\psi_{nt}(x,y,\tau)\widehat{\psi}(x,y,\tau)+\phi_{nt}(x,y,\tau)\widehat{\phi}(x,y,\tau)]+\rho_2\omega_{nt}(x,y,\tau)\widehat{\omega}(x,y,\tau)\}\mathrm{d}x\mathrm{d}y\\&\longrightarrow\int_\Omega\{\rho_1[\psi_t^*(x,y,\tau)\widehat{\psi}(x,y,\tau)+\phi_t^*(x,y,\tau)\widehat{\phi}(x,y,\tau)]+\rho_2\omega_t^*(x,y,\tau)\widehat{\omega}(x,y,\tau)\}\mathrm{d}x\mathrm{d}y.\end{aligned}\tag{3.7}$$

因此，当 $U=U^*$ 时，对 (3.5) 和 (3.7) 式，令 $n\to\infty$ 取极限得, Φ^* 是系统 (2.2) 的弱解. 从而，对任意控制 $U=(u_1,u_2,u_3)\in L^2(0,T;\mathcal{U})$ 到 $(\Phi,\Phi_t)\in L^\infty([0,T];\mathcal{H})$ 的映射是连续的. 又由目标函数的定义可知，目标函数 $J_T(\mathcal{Y}_0,U)$ 是弱下半连续的，这蕴含着

$$0\le J_T(\mathcal{Y}_0,U^*)\le\lim_{n\to\infty}\inf_{U\in L^2(0,T;\mathcal{U})}J_T(\mathcal{Y}_0,U_n)=\theta<\infty.$$

所以, $(\mathcal{Y}^*,U^*)$ 是最优问题 (2.1) 满足系统 (2.2) 的最优解. 由于 $J_T(\mathcal{Y}_0,U)$ 是严格凸的，于是最优解是唯一的. 证毕. ▎

4 最优性条件

定理 4.1 考虑如下线性系统

$$\begin{cases}\rho_1\psi_{tt}-D(\psi_{xx}+\dfrac{1-\mu}{2}\psi_{yy}+\dfrac{1+\mu}{2}\phi_{xy})+K(\psi+\omega_x)=0,(x,y,t)\in\Omega\times(0,T),\\ \rho_1\phi_{tt}-D(\phi_{yy}+\dfrac{1-\mu}{2}\phi_{xx}+\dfrac{1+\mu}{2}\psi_{xy})+K(\phi+\omega_y)=0,(x,y,t)\in\Omega\times(0,T),\\ \rho_2\omega_{tt}-K[(\psi+\omega_x)_x+(\phi+\omega_y)_y]=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \psi=\phi=\omega=0,\quad(x,y,t)\in\Gamma_0\times(0,T),\\ D[\nu_1\psi_x+\mu\nu_1\phi_y+\dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_2]=u_1,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ D[\nu_2\phi_y+\mu\nu_2\psi_x+\dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_1]=u_2,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ K(\dfrac{\partial\omega}{\partial\nu}+\nu_1\psi+\nu_2\phi)=u_3,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ (\psi(x,y,0),\phi(x,y,0),\omega(x,y,0))=(0,0,0),\quad(x,y)\in\Omega,\\ (\psi_t(x,y,0),\phi_t(x,y,0),\omega_t(x,y,0))=(0,0,0),\quad(x,y)\in\Omega\end{cases}\tag{4.1}$$

与对偶系统

$$\begin{cases}\rho_1 p_{tt}-D(p_{xx}+\dfrac{1-\mu}{2}p_{yy}+\dfrac{1+\mu}{2}q_{xy})+K(p+r_x)=g_1,(x,y,t)\in\Omega\times(0,T),\\ \rho_1 q_{tt}-D(q_{yy}+\dfrac{1-\mu}{2}q_{xx}+\dfrac{1+\mu}{2}p_{xy})+K(q+r_y)=g_2,(x,y,t)\in\Omega\times(0,T),\\ \rho_2 r_{tt}-K[(p+r_x)_x+(q+r_y)_y]=g_3,\quad (x,y,t)\in\Omega\times(0,T),\\ \psi=\phi=\omega=0,\quad (x,y,t)\in\Gamma_0\times(0,T),\\ D[\nu_1 p_x+\mu\nu_1 q_y+\dfrac{1-\mu}{2}(p_y+q_x)\nu_2]=0,\quad (x,y,t)\in\Gamma_1\times(0,T),\\ D[\nu_2 q_y+\mu\nu_2 p_x+\dfrac{1-\mu}{2}(p_y+q_x)\nu_1]=0,\quad (x,y,t)\in\Gamma_1\times(0,T),\\ K(\dfrac{\partial r}{\partial\nu}+\nu_1 p+\nu_2 q)=0,\quad (x,y,t)\in\Gamma_1\times(0,T),\\ (p(x,y,T),q(x,y,T),r(x,y,T))=(p_0(T),q_0(T),r_0(T)),\quad (x,y)\in\Omega,\\ (p_t(x,y,T),q_t(x,y,T),r_t(x,y,T))=(p_1(T),q_1(T),r_1(T)),\quad (x,y)\in\Omega,\end{cases}\tag{4.2}$$

其中 $(g_1,g_2,g_3)\in L^2([0,T];W^*)$，$((p_0(T),q_0(T),r_0(T));(p_1(T),q_1(T),r_1(T)))\in H^*\times W^*$，$H^*,W^*$ 分别表示 H,W 的对偶空间．且成立着如下的等式

$$\begin{aligned}&\int_0^T\int_{\Gamma_1}(pu_1+qu_2+ru_3)\mathrm{d}\Gamma_1\mathrm{d}t\\ =&\int_0^T\langle(\psi,\phi,\omega),(g_1,g_2,g_3)\rangle_{W\times W^*}\mathrm{d}t+((p_0(T),q_0(T),r_0(T)),(\psi_t(T),\phi_t(T),\omega_t(T)))_H\\ &-\langle(\rho_1\psi(T),\rho_1\phi(T),\rho_2\omega(T)),(p_1(T),q_1(T),r_1(T))\rangle_{W\times W^*}.\end{aligned}\tag{4.3}$$

证 对于 $(p,q,r)\in C^0([0,T];W)\cap C^1([0,T];H)$，并用 (p,q,r) 分别乘以系统 (4.1) 前三式，在 $[0,T]\times\Omega$ 上积分并相加得

$$\begin{aligned}&\int_0^T\int_\Omega[\rho_1(\psi_{tt}p+\phi_{tt}q)+\rho_2\omega_{tt}r]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ =&\int_0^T\int_\Omega[L_1\{\psi,\phi,\omega\}p+L_2\{\psi,\phi,\omega\}q+L_3\{\psi,\phi,\omega\}r]\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned}$$

分部积分得

$$\begin{aligned}&\int_\Omega[\rho_1((\psi_t p-\psi p_t)+(\phi_t q-\phi q_t))+\rho_2(\omega_t r-\omega r_t)]_0^T\,\mathrm{d}x\mathrm{d}y\\ &+\int_0^T\int_\Omega[\rho_1(\psi p_{tt}+\phi q_{tt})+\rho_2\omega r_{tt}]\mathrm{d}x\mathrm{d}y\mathrm{d}t+\int_0^T a(\psi,\phi,\omega;p,q,r)\mathrm{d}t\\ =&\int_0^T\int_\Omega[L_1\{\psi,\phi,\omega\}p+L_2\{\psi,\phi,\omega\}q+L_3\{\psi,\phi,\omega\}r]\mathrm{d}x\mathrm{d}y\mathrm{d}t,\end{aligned}$$

即

$$\begin{aligned}&\int_\Omega[\rho_1((\psi_t(T)p_0(T)-\psi(T)p_1(T))+(\phi_t(T)q_0(T)-\phi(T)q_1(T))\\ &+\rho_2(\omega_t(T)r_0(T)-\omega(T)r_1(T))]\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega[\rho_1(\psi p_{tt}+\phi q_{tt})+\rho_2\omega r_{tt}]\mathrm{d}x\mathrm{d}y\mathrm{d}t+\int_0^T a(\psi,\phi,\omega;p,q,r)\mathrm{d}t\\
=&\int_0^T\int_{\Gamma_1}[u_1p+u_2q+u_3r]\mathrm{d}\Gamma_1\mathrm{d}t.
\end{aligned}
$$

进一步有

$$
\begin{aligned}
&\int_\Omega[\rho_1((\psi_t(T)p_0(T)-\psi(T)p_1(T))+(\phi_t(T)q_0(T)-\phi(T)q_1(T))\\
&+\rho_2(\omega_t(T)r_0(T)-\omega(T)r_1(T))]\mathrm{d}x\mathrm{d}y+\int_0^T\int_\Omega[\rho_1(\psi p_{tt}+\phi q_{tt})+\rho_2\omega r_{tt}]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&+\int_0^T\int_{\Gamma_1}(\psi F_1\{p,q\}+\phi F_2\{p,q\}+\omega F_3\{p,q,r\})\mathrm{d}\Gamma_1\mathrm{d}t\\
=&\int_0^T\left\{\int_\Omega[\psi L_1\{p,q,r\}+\phi L_2\{p,q,r\}+\omega L_3\{p,q,r\}]\mathrm{d}x\mathrm{d}y+\int_{\Gamma_1}[u_1p+u_2q+u_3r]\mathrm{d}\Gamma_1\right\}\mathrm{d}t.
\end{aligned}
$$

因此有

$$
\begin{aligned}
&\int_0^T\int_{\Gamma_1}(pu_1+qu_2+ru_3)\mathrm{d}\Gamma_1\mathrm{d}t\\
=&\int_0^T\langle(\psi,\phi,\omega),(g_1,g_2,g_3)\rangle_{W\times W^*}\mathrm{d}t+((p_0(T),q_0(T),r_0(T)),(\psi_t(T),\phi_t(T),\omega_t(T)))_H\\
&-\langle(\rho_1\psi(T),\rho_1\phi(T),\rho_2\omega(T)),(p_1(T),q_1(T),r_1(T))\rangle_{W\times W^*}.
\end{aligned}
$$

由能量估计式及线性双曲方程关于时间的可逆性知系统 (4.2) 的解 $(p,q,r)\in C^0([0,T];H^*)\cap C^1([0,T];W^*)$, 而 $W,H,L^2([0,T];H)$ 分别在 $H^*,W^*,L^2([0,T];W^*)$ 中稠密，由稠密性理论得

$$(p,q,r)\in C^0([0,T];H)\cap C^1([0,T];W).$$

证毕. ▌

定理 4.2 若 $(\overline{\Phi};\overline{U})=((\overline{\psi},\overline{\phi},\overline{\omega});(\overline{u}_1,\overline{u}_2,\overline{u}_3))$ 是最优控制问题 (OCP) 的最优解，则满足以下最优性条件

$$
\begin{cases}
\rho_1\overline{\psi}_{tt}-D(\overline{\psi}_{xx}+\frac{1-\mu}{2}\overline{\psi}_{yy}+\frac{1+\mu}{2}\overline{\phi}_{xy})+K(\overline{\psi}+\overline{\omega}_x)=0,(x,y,t)\in\Omega\times(0,T),\\
\rho_1\overline{\phi}_{tt}-D(\overline{\phi}_{yy}+\frac{1-\mu}{2}\overline{\phi}_{xx}+\frac{1+\mu}{2}\overline{\psi}_{xy})+K(\overline{\phi}+\overline{\omega}_y)=0,(x,y,t)\in\Omega\times(0,T),\\
\rho_2\overline{\omega}_{tt}-K[(\overline{\psi}+\overline{\omega}_x)_x+(\overline{\phi}+\overline{\omega}_y)_y]=0,\quad(x,y,t)\in\Omega\times(0,T),\\
\overline{\psi}=\overline{\phi}=\overline{\omega}=0,\quad(x,y,t)\in\Gamma_0\times(0,T),\\
D[\nu_1\overline{\psi}_x+\mu\nu_1\overline{\phi}_y+\frac{1-\mu}{2}(\overline{\psi}_y+\overline{\phi}_x)\nu_2]=\overline{u}_1,\quad(x,y,t)\in\Gamma_1\times(0,T),\\
D[\nu_2\overline{\phi}_y+\mu\nu_2\overline{\psi}_x+\frac{1-\mu}{2}(\overline{\psi}_y+\overline{\phi}_x)\nu_1]=\overline{u}_2,\quad(x,y,t)\in\Gamma_1\times(0,T),\\
K(\frac{\partial\overline{\omega}}{\partial\nu}+\nu_1\overline{\psi}+\nu_2\overline{\phi})=\overline{u}_3,\quad(x,y,t)\in\Gamma_1\times(0,T),\\
(\overline{\psi}(x,y,0),\overline{\phi}(x,y,0),\overline{\omega}(x,y,0))=(\psi_{01},\phi_{01},\omega_{01}),\quad(x,y)\in\Omega,\\
(\overline{\psi}_t(x,y,0),\overline{\phi}_t(x,y,0),\overline{\omega}_t(x,y,0))=(\psi_{02},\phi_{02},\omega_{02}),\quad(x,y)\in\Omega
\end{cases}
\tag{4.4}
$$

和

$$
\begin{cases}
\rho_1\overline{p}_{tt}-D(\overline{p}_{xx}+\dfrac{1-\mu}{2}\overline{p}_{yy}+\dfrac{1+\mu}{2}\overline{q}_{xy})+K(\overline{p}+\overline{r}_x)\\
=-[\rho_1\overline{\psi}_{tt}+D(\overline{\psi}_{xx}+\dfrac{1-\mu}{2}\overline{\psi}_{yy}+\dfrac{1+\mu}{2}\overline{\phi}_{xy})-K(\overline{\psi}+\overline{\omega}_x)],(x,y,t)\in\Omega\times(0,T),\\
\rho_1\overline{q}_{tt}-D(\overline{q}_{yy}+\dfrac{1-\mu}{2}\overline{q}_{xx}+\dfrac{1+\mu}{2}\overline{p}_{xy})+K(\overline{q}+\overline{r}_y)\\
=-[\rho_1\overline{\phi}_{tt}+D(\overline{\phi}_{yy}+\dfrac{1-\mu}{2}\overline{\phi}_{xx}+\dfrac{1+\mu}{2}\overline{\psi}_{xy})-K(\overline{\phi}+\overline{\omega}_y)],(x,y,t)\in\Omega\times(0,T),\\
\rho_2\overline{r}_{tt}-K[(\overline{p}+\overline{r}_x)_x+(\overline{q}+\overline{r}_y)_y]\\
=-(\rho_2\overline{\omega}_{tt}+K[(\overline{\psi}+\overline{\omega}_x)_x+(\overline{\phi}+\overline{\omega}_y)_y]),\quad (x,y,t)\in\Omega\times(0,T),\\
\overline{p}=\overline{q}=\overline{r}=0,\quad (x,y,t)\in\Gamma_0\times(0,T),\\
D[\nu_1\overline{p}_x+\mu\nu_1\overline{q}_y+\dfrac{1-\mu}{2}(\overline{p}_y+\overline{q}_x)\nu_2]=0,\quad (x,y,t)\in\Gamma_1\times(0,T),\\
D[\nu_2\overline{q}_y+\mu\nu_2\overline{p}_x+\dfrac{1-\mu}{2}(\overline{p}_y+\overline{q}_x)\nu_1]=0,\quad (x,y,t)\in\Gamma_1\times(0,T),\\
K(\dfrac{\partial\overline{r}}{\partial\nu}+\nu_1\overline{p}+\nu_2\overline{q})=0,\quad (x,y,t)\in\Gamma_1\times(0,T),\\
(\overline{p}(x,y,T),\overline{q}(x,y,T),\overline{\omega}(x,y,T))=(0,0,0),\quad (x,y)\in\Omega,\\
(\overline{p}_t(x,y,T),\overline{q}_t(x,y,T),\overline{r}_t(x,y,T))=-(\overline{\psi}_t(T),\overline{\phi}_t(T),\overline{\omega}_t(T)),\quad (x,y)\in\Omega.
\end{cases}
\tag{4.5}
$$

其中 $(\overline{p},\overline{q},\overline{r})\in C^0([0,T];H^*)\cap C^1([0,T];W^*)$ 是对偶系统 (4.2) 的解，且在 $[0,T]\times\Gamma_1$ 上有

$$
(\overline{p},\overline{q},\overline{r})=-\beta(\overline{u}_1,\overline{u}_2,\overline{u}_3).
\tag{4.6}
$$

证 由目标函数的表达式得

$$
\begin{aligned}
J_T(\mathcal{Y}_0,U)&=\int_0^T(\frac{1}{2}\|\mathcal{Y}(t)\|^2_{\mathcal{H}}+\frac{\beta}{2}\|U(t)\|^2_{\mathcal{U}})\mathrm{d}t\\
&=\frac{1}{2}\int_0^T a(\psi,\phi,\omega;\psi,\phi,\omega)\mathrm{d}t+\frac{1}{2}\int_0^T\int_\Omega[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\quad+\frac{\beta}{2}\int_0^T\int_{\Gamma_1}(|u_1|^2+|u_2|^2+|u_3|^2)\mathrm{d}\Gamma_1\mathrm{d}t.
\end{aligned}
\tag{4.7}
$$

对于任意的 $(\delta u_1,\delta u_2,\delta u_3)\in L^2([0,T];\mathcal{U})$, 对目标函数作变分得

$$
\begin{aligned}
&\frac{\partial J_T(\mathcal{Y}_0,U)}{\partial u_1}\delta u_1+\frac{\partial J_T(\mathcal{Y}_0,U)}{\partial u_2}\delta u_2+\frac{\partial J_T(\mathcal{Y}_0,U)}{\partial u_3}\delta u_3\\
=&\int_0^T a(\psi,\phi,\omega;\delta\psi,\delta\phi,\delta\omega)\mathrm{d}t+\int_0^T\int_\Omega[\rho_1(\psi_t\delta\psi_t+\phi_t\delta\phi_t)+\rho_2\omega_t\delta\omega_t]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&+\beta\int_0^T\int_{\Gamma_1}(u_1\delta u_1+u_2\delta u_2+u_3\delta u_3)\mathrm{d}\Gamma_1\mathrm{d}t.
\end{aligned}
$$

分部积分得

$$
\frac{\partial J_T(\mathcal{Y}_0,U)}{\partial u_1}\delta u_1+\frac{\partial J_T(\mathcal{Y}_0,U)}{\partial u_2}\delta u_2+\frac{\partial J_T(\mathcal{Y}_0,U)}{\partial u_3}\delta u_3
$$

$$
\begin{aligned}
&= -\int_0^T\int_\Omega[\delta\psi(\rho_1\psi_{tt}+L_1\{\psi,\phi,\omega\})+\delta\phi(\rho_1\phi_{tt}+L_2\{\psi,\phi,\omega\})+\delta\omega(\rho_2\omega_{tt}+L_3\{\psi,\phi,\omega\})]\mathrm{d}t\\
&\quad+\int_\Omega[\rho_1(\delta\psi(T)\psi_t(T)+\delta\phi(T)\phi_t(T))+\rho_2\delta\omega(T)\omega_t(T)]\mathrm{d}x\mathrm{d}y\\
&\quad+\beta\int_0^T\int_{\Gamma_1}(u_1\delta u_1+u_2\delta u_2+u_3\delta u_3)\mathrm{d}\Gamma_1\mathrm{d}t,
\end{aligned}
$$

其中 $(\delta\psi,\delta\phi,\delta\omega)\in C^0([0,T];W)\cap C^1([0,T];H)$ 是以下系统的弱解

$$
\begin{cases}
\rho_1\delta\psi_{tt}-D((\delta\psi)_{xx}+\frac{1-\mu}{2}(\delta\psi)_{yy}+\frac{1+\mu}{2}(\delta\phi)_{xy})+K(\delta\psi+(\delta\omega)_x)=0,\\
\qquad\qquad (x,y,t)\in\Omega\times(0,T),\\
\rho_1\delta\phi_{tt}-D((\delta\phi)_{yy}+\frac{1-\mu}{2}(\delta\phi)_{xx}+\frac{1+\mu}{2}(\delta\psi)_{xy})+K(\delta\phi+(\delta\omega)_y)=0,\\
\qquad\qquad (x,y,t)\in\Omega\times(0,T),\\
\rho_2\delta\omega_{tt}-K[(\delta\psi+(\delta\omega)_x)_x+(\delta\phi+(\delta\omega)_y)_y]=0,\quad (x,y,t)\in\Omega\times(0,T),\\
\delta\psi=\delta\phi=\delta\omega=0,\quad (x,y,t)\in\Gamma_0\times(0,T),\\
D[\nu_1(\delta\psi)_x+\mu\nu_1(\delta\phi)_y+\frac{1-\mu}{2}((\delta\psi)_y+(\delta\phi)_x)\nu_2]=\delta u_1,(x,y,t)\in\Gamma_1\times(0,T),\\
D[\nu_2(\delta\phi)_y+\mu\nu_2(\delta\psi)_x+\frac{1-\mu}{2}((\delta\psi)_y+(\delta\phi)_x)\nu_1]=\delta u_2,(x,y,t)\in\Gamma_1\times(0,T),\\
K(\frac{\partial(\delta\omega)}{\partial\nu}+\nu_1\delta\psi+\nu_2\delta\phi)=\delta u_3,\quad (x,y,t)\in\Gamma_1\times(0,T),\\
(\delta\psi(x,y,0),\delta\phi(x,y,0),\delta\omega(x,y,0))=(0,0,0),\quad (x,y)\in\Omega,\\
(\delta\psi_t(x,y,0),\delta\phi_t(x,y,0),\delta\omega_t(x,y,0))=(0,0,0),\quad (x,y)\in\Omega.
\end{cases}
\tag{4.8}
$$

则一阶最优性条件等价于

$$
\begin{aligned}
&-\int_0^T\int_\Omega[\delta\psi(\rho_1\psi_{tt}+L_1\{\psi,\phi,\omega\})+\delta\phi(\rho_1\phi_{tt}+L_2\{\psi,\phi,\omega\})+\delta\omega(\rho_2\omega_{tt}+L_3\{\psi,\phi,\omega\})]\mathrm{d}t\\
&+\int_\Omega[\rho_1(\delta\psi(T)\psi_t(T)+\delta\phi(T)\phi_t(T))+\rho_2\delta\omega(T)\omega_t(T)]\mathrm{d}x\mathrm{d}y\\
&+\beta\int_0^T\int_{\Gamma_1}(u_1\delta u_1+u_2\delta u_2+u_3\delta u_3)\mathrm{d}\Gamma_1\mathrm{d}t=0.
\end{aligned}
\tag{4.9}
$$

另一方面，由定理 4.1 得

$$
\begin{aligned}
&\int_0^T\int_{\Gamma_1}(p\delta u_1+q\delta u_2+r\delta u_3)\mathrm{d}\Gamma_1\mathrm{d}t\\
&=\int_0^T\langle(\delta\psi,\delta\phi,\delta\omega),(g_1,g_2,g_3)\rangle_{W\times W^*}\mathrm{d}t+((p_0(T),q_0(T),r_0(T)),(\delta\psi_t(T),\delta\phi_t(T),\delta\omega_t(T)))_H\\
&\quad-\langle(\rho_1\delta\psi(T),\rho_1\delta\phi(T),\rho_2\delta\omega(T)),(p_1(T),q_1(T),r_1(T))\rangle_{W\times W^*}.
\end{aligned}
\tag{4.10}
$$

比较 (4.9) 式和 (4.10) 式，并由 $(\delta u_1,\delta u_2,\delta u_3)\in L^2([0,T];\mathcal{U})$ 的任意性，我们可以得到

$$
g_1=-[\rho_1\psi_{tt}+D(\psi_{xx}+\frac{1-\mu}{2}\psi_{yy}+\frac{1+\mu}{2}\phi_{xy})-K(\psi+\omega_x)],
$$

$$g_2=-[\rho_1\phi_{tt}+D(\phi_{yy}+\frac{1-\mu}{2}\phi_{xx}+\frac{1+\mu}{2}\psi_{xy})-K(\phi+\omega_y)],$$
$$g_3=-[\rho_2\omega_{tt}+K((\psi+\omega_x)_x+(\phi+\omega_y)_y)]$$
$$p_0(T)=q_0(T)=r_0(T)=0,$$
$$p_1(T)=-\psi_t(T),q_1(T)=-\phi_t(T),r_1(T)=-\omega_t(T),$$
$$(p,q,r)=-\beta(u_1,u_2,u_3),(x,y,t)\in[0,T]\times\Gamma_1.$$

即由变分原理所得到的最优解满足 (4.4)–(4.6) 式. 证毕. ∎

5 能观性与能量指数衰减的等价性

本节我们主要讨论 Mindlin-Timoshenko 板系统的能观性与系统能量指数衰减之间的关系. 首先我们采用文献 [21] 的方法证明如下的结果.

引理 5.1 假定存在 $(x_0,y_0)\in\mathbb{R}^2$, 满足 $\widetilde{\Gamma}=\{(x,y)\in\Gamma_1|(x-x_0,y-y_0)\cdot\nu>0\}$ 非空, 则存在 $T_1>0$, 使得对任意 $T\geq T_1$ 有

$$C_1\|\mathcal{Y}_0\|_{\mathcal{H}}^2\leq\int_0^T\int_{\widetilde{\Gamma}}(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}\widetilde{\Gamma}\mathrm{d}t,\tag{5.1}$$

存在 $T_2>0$, 使得对任意 $T\geq T_2$ 有

$$C_2\|\mathcal{Y}_0\|_{\mathcal{H}}^2\leq\int_0^T\int_{\Omega}(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}x\mathrm{d}y\mathrm{d}t,\tag{5.2}$$

其中, C_1,C_2 均为仅与 T 有关的正常数, $\widetilde{\Phi}=(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})$ 是如下齐次系统的弱解, 且 $(\widetilde{\Phi},\widetilde{\Phi}_t)\in C^0([0,T];\mathcal{H})$, 有

$$\begin{cases}\rho_1\widetilde{\psi}_{tt}-D(\widetilde{\psi}_{xx}+\frac{1-\mu}{2}\widetilde{\psi}_{yy}+\frac{1+\mu}{2}\widetilde{\phi}_{xy})+K(\widetilde{\psi}+\widetilde{\omega}_x)=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \rho_1\widetilde{\phi}_{tt}-D(\widetilde{\phi}_{yy}+\frac{1-\mu}{2}\widetilde{\phi}_{xx}+\frac{1+\mu}{2}\widetilde{\psi}_{xy})+K(\widetilde{\phi}+\widetilde{\omega}_y)=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \rho_2\widetilde{\omega}_{tt}-K[(\widetilde{\psi}+\widetilde{\omega}_x)_x+(\widetilde{\phi}+\widetilde{\omega}_y)_y]=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \widetilde{\psi}=\widetilde{\phi}=\widetilde{\omega}=0,\quad(x,y,t)\in\Gamma_0\times(0,T),\\ D[\nu_1\widetilde{\psi}_x+\mu\nu_1\widetilde{\phi}_y+\frac{1-\mu}{2}(\widetilde{\psi}_y+\widetilde{\phi}_x)\nu_2]=0,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ D[\nu_2\widetilde{\phi}_y+\mu\nu_2\widetilde{\psi}_x+\frac{1-\mu}{2}(\widetilde{\psi}_y+\widetilde{\phi}_x)\nu_1]=0,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ K(\frac{\partial\widetilde{\omega}}{\partial\nu}+\nu_1\widetilde{\psi}+\nu_2\widetilde{\phi})=0,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ (\widetilde{\psi}(x,y,0),\widetilde{\phi}(x,y,0),\widetilde{\omega}(x,y,0))=(\psi_{01},\phi_{01},\omega_{01}),\quad(x,y)\in\Omega,\\ (\widetilde{\psi}_t(x,y,0),\widetilde{\phi}_t(x,y,0),\widetilde{\omega}_t(x,y,0))=(\psi_{02},\phi_{02},\omega_{02}),\quad(x,y)\in\Omega.\end{cases}\tag{5.3}$$

证 记 $F(x,y)=\eta(x,y)((x-x_0),(y-y_0))$, 其中, 乘子 $\eta(\cdot)\in C^1(\overline{\Omega})$ 且满足在 $\overline{\Gamma}_0$ 上, $\eta=0$; 在 Γ_1 上, $\eta=1$. 对边界 Γ_1 作如下分割

$$\widetilde{\Gamma}_1=\{(x,y)\in\Gamma_1|F(x,y)\cdot\nu>0\},$$

$$\widehat{\Gamma}_1=\Gamma_1\backslash\widetilde{\Gamma}_1=\{(x,y)\in\Gamma_1|F(x,y)\cdot\nu\leq 0\}.$$

记 $M^2=\max\limits_{(x,y)\in\overline{\Omega}}|F(x,y)|^2=\max\limits_{(x,y)\in\overline{\Omega}}\{|x-x_0|^2+|y-y_0|^2\}$, $2M_1=\max\{\rho_1,\rho_2\}$. 将 $F\cdot\nabla\widetilde{\psi},F\cdot\nabla\widetilde{\phi},F\cdot\nabla\widetilde{\omega}$ 分别乘以系统 (5.3) 前三式，并在 $[0,T]\times\Omega$ 上积分，则有

$$\begin{aligned}&\int_0^T\int_\Omega[\rho_1(\widetilde{\psi}_{tt}F\cdot\nabla\widetilde{\psi}+\widetilde{\phi}_{tt}F\cdot\nabla\widetilde{\phi})+\rho_2\widetilde{\omega}_{tt}F\cdot\nabla\widetilde{\omega}]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\=&\int_0^T\int_\Omega[L_1\{\widetilde{\psi},\widetilde{\phi},\widetilde{\omega}\}F\cdot\nabla\widetilde{\psi}+L_2\{\widetilde{\psi},\widetilde{\phi},\widetilde{\omega}\}F\cdot\nabla\widetilde{\phi})+L_3\{\widetilde{\psi},\widetilde{\phi},\widetilde{\omega}\}F\cdot\nabla\widetilde{\omega}]\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned}$$

分部积分得

$$\begin{aligned}&\int_0^T\int_{\Gamma_1}(F\cdot\nu)[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}\Gamma_1\mathrm{d}t\\=&2\int_\Omega[\rho_1(\widetilde{\psi}_tF\cdot\nabla\widetilde{\psi}+\widetilde{\phi}_tF\cdot\nabla\widetilde{\phi})+\rho_2\widetilde{\omega}_tF\cdot\nabla\widetilde{\omega}]_0^T\mathrm{d}x\mathrm{d}y\\&+\int_0^T\int_\Omega\mathrm{div}F[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t+2\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t\\&+D\int_0^T\int_{\Gamma_1}(F\cdot\nu)[|\widetilde{\psi}_x|^2+|\widetilde{\phi}_y|^2+2\mu\widetilde{\psi}_x\widetilde{\phi}_y+\frac{1-\mu}{2}|\widetilde{\psi}_y+\widetilde{\phi}_x|^2]\mathrm{d}\Gamma_1\mathrm{d}t\\&+K\int_0^T\int_{\Gamma_1}(F\cdot\nu)[|\widetilde{\psi}+\widetilde{\omega}_x|^2+|\widetilde{\phi}+\widetilde{\omega}_y|^2]\mathrm{d}\Gamma_1\mathrm{d}t\\&-D\int_0^T\int_\Omega\mathrm{div}F[|\widetilde{\psi}_x|^2+|\widetilde{\phi}_y|^2+2\mu\widetilde{\psi}_x\widetilde{\phi}_y+\frac{1-\mu}{2}|\widetilde{\psi}_y+\widetilde{\phi}_x|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\&-K\int_0^T\int_\Omega\mathrm{div}F[|\widetilde{\psi}+\widetilde{\omega}_x|^2+|\widetilde{\phi}+\widetilde{\omega}_y|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t,\end{aligned}$$

即

$$\begin{aligned}&\int_0^T\int_{\Gamma_1}(F\cdot\nu)[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}\Gamma_1\mathrm{d}t\\=&2\int_\Omega[\rho_1(\widetilde{\psi}_tF\cdot\nabla\widetilde{\psi}+\widetilde{\phi}_tF\cdot\nabla\widetilde{\phi})+\rho_2\widetilde{\omega}_tF\cdot\nabla\widetilde{\omega}]_0^T\mathrm{d}x\mathrm{d}y\\&+2\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t+2\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t\\&-2D\int_0^T\int_\Omega[|\widetilde{\psi}_x|^2+|\widetilde{\phi}_y|^2+2\mu\widetilde{\psi}_x\widetilde{\phi}_y+\frac{1-\mu}{2}|\widetilde{\psi}_y+\widetilde{\phi}_x|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\&-2K\int_0^T\int_\Omega[|\widetilde{\psi}+\widetilde{\omega}_x|^2+|\widetilde{\phi}+\widetilde{\omega}_y|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned}\tag{5.4}$$

由 Poincaré 不等式得

$$\begin{aligned}&\left|\int_\Omega[\rho_1(\widetilde{\psi}_tF\cdot\nabla\widetilde{\psi}+\widetilde{\phi}_tF\cdot\nabla\widetilde{\phi})+\rho_2\widetilde{\omega}_tF\cdot\nabla\widetilde{\omega}]_0^T\mathrm{d}x\mathrm{d}y\right|\\\leq&c_1\left\{\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]_0^T\mathrm{d}x\mathrm{d}y+a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})|_0^T\right\}\\\leq&2c_1I(0).\end{aligned}\tag{5.5}$$

将 $\widetilde{\psi},\widetilde{\phi},\widetilde{\omega}$ 分别乘以系统 (4.3) 前三式，并在 $[0,T]\times\Omega$ 上积分并相加，得

$$\begin{aligned}&\int_0^T\int_\Omega[\rho_1(\widetilde{\psi}_{tt}\widetilde{\psi}+\widetilde{\phi}_{tt}\widetilde{\phi})+\rho_2\widetilde{\omega}_{tt}\widetilde{\omega}]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\=&\int_0^T\int_\Omega[L_1\{\widetilde{\psi},\widetilde{\phi},\widetilde{\omega}\}\widetilde{\psi}+L_2\{\widetilde{\psi},\widetilde{\phi},\widetilde{\omega}\}\widetilde{\phi}+L_3\{\widetilde{\psi},\widetilde{\phi},\widetilde{\omega}\}\widetilde{\omega}]\mathrm{d}x\mathrm{d}y\mathrm{d}t,\end{aligned}$$

分部积分得

$$\begin{aligned}&\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t-\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t\\=&\int_\Omega[\rho_1(\widetilde{\psi}_t\widetilde{\psi}+\widetilde{\phi}_t\widetilde{\phi})+\rho_2(\widetilde{\omega}_t\widetilde{\omega}]_0^T\mathrm{d}x\mathrm{d}y.\end{aligned}\tag{5.6}$$

再由 Poincaré 不等式得

$$\left|\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t-\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t\right|\le c_0I(0).\tag{5.7}$$

由 (5.4), (5.5) 和 (5.7) 式得

$$\begin{aligned}MM_1\int_0^T\int_{\widetilde{\Gamma}}[|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2]\mathrm{d}\widetilde{\Gamma}\mathrm{d}t&\ge\frac{1}{2}\int_0^T\int_{\Gamma_1}(F\cdot\nu)[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}\Gamma_1\mathrm{d}t\\&\ge\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t-2c_1I(0)-c_0I(0).\end{aligned}\tag{5.8}$$

又由 (5.8) 式得

$$\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t\ge\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t-c_0I(0).\tag{5.9}$$

因此，由 (5.8)–(5.9) 式得

$$\begin{aligned}&MM_1\int_0^T\int_{\widetilde{\Gamma}}[|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2]\mathrm{d}\widetilde{\Gamma}\mathrm{d}t\\\ge&\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t-2c_1I(0)-2c_0I(0).\end{aligned}\tag{5.10}$$

由 (5.8) 和 (5.10) 式可得

$$\begin{aligned}&MM_1\int_0^T\int_{\widetilde{\Gamma}}[|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2]\mathrm{d}\widetilde{\Gamma}\mathrm{d}t\\\ge&\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t+\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t-4c_1I(0)-3c_0I(0)\\\ge&TI(0)-4c_1I(0)-3c_0I(0).\end{aligned}$$

因此，记 $T_1=4c_1+3c_0>0$, 当 $T\ge T_1$ 时，记 $C_1=\frac{1}{2MM_0}(T-T_1)\ge0$, 则有

$$C_1\|\mathcal{Y}_0\|_{\mathcal{H}}^2=C_1I(0)\le\int_0^T\int_{\widetilde{\Gamma}}(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}\widetilde{\Gamma}\mathrm{d}t.$$

故 (5.1) 式得证.

又由 (5.7) 式得

$$\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\geq\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t-c_0I(0).$$

故

$$\begin{aligned}&2\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ \geq&\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t+\int_0^T a(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega};\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})\mathrm{d}t-c_0I(0),\end{aligned}$$

即

$$2\int_0^T\int_\Omega[\rho_1(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2)+\rho_2|\widetilde{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\geq TI(0)-c_0I(0).\tag{5.11}$$

因此，记 $T_2=c_0>0$ 当 $T\geq T_2$ 时，记 $C_2=\frac{1}{2M_1}(T-T_2)\geq0$, 则由 (5.11) 式得

$$C_2\|\mathcal{Y}_0\|_{\mathcal{H}}^2=C_2I(0)\leq\int_0^T\int_\Omega(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

从而，(5.2) 式得证．引理 5.1 证毕. ▮

定理 5.1 假定 $\mathcal{Y}_0\in\mathcal{H}$, 系统 (1.1)–(1.2) 能量依 $\mathcal{H}$ 范数一致指数收敛到 0, 即存在与 $\mathcal{Y}_0$ 无关，而与 T_1 有关的正常数 M,α 使得

$$\|\mathcal{Y}\|_{\mathcal{H}}^2\leq M\mathrm{e}^{-\alpha t}\|\mathcal{Y}_0\|_{\mathcal{H}}^2$$

当且仅当能观性条件 (5.1) 成立.

证 (I) 充分性

首先闭环系统 (1.1) 和 (1.2) 是适定的，且其唯一弱解为

$$\Phi=(\psi,\phi,\omega)\in C^0([0,\infty);W)\cap C^1([0,\infty);H),$$

对于任意 $T>0$, 考虑如下控制系统

$$\begin{cases}\rho_1\psi_{tt}-D(\psi_{xx}+\dfrac{1-\mu}{2}\psi_{yy}+\dfrac{1+\mu}{2}\phi_{xy})+K(\psi+\omega_x)=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \rho_1\phi_{tt}-D(\phi_{yy}+\dfrac{1-\mu}{2}\phi_{xx}+\dfrac{1+\mu}{2}\psi_{xy})+K(\phi+\omega_y)=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \rho_2\omega_{tt}-K[(\psi+\omega_x)_x+(\phi+\omega_y)_y]=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \psi=\phi=\omega=0,\quad(x,y,t)\in\Gamma_0\times(0,T),\\ D[\nu_1\psi_x+\mu\nu_1\phi_y+\dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_2]=-\psi_t,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ D[\nu_2\phi_y+\mu\nu_2\psi_x+\dfrac{1-\mu}{2}(\psi_y+\phi_x)\nu_1]=-\phi_t,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ K(\dfrac{\partial\omega}{\partial\nu}+\nu_1\psi+\nu_2\phi)=-\omega_t,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ (\psi(x,y,0),\phi(x,y,0),\omega(x,y,0))=(\psi_{01},\phi_{01},\omega_{01}),\quad(x,y)\in\Omega,\\ (\psi_t(x,y,0),\phi_t(x,y,0),\omega_t(x,y,0))=(\psi_{02},\phi_{02},\omega_{02}),\quad(x,y)\in\Omega.\end{cases}\tag{5.12}$$

用 ψ_t, ϕ_t, ω_t 分别乘以系统 (5.12) 前三式，并在 $[0,T_1]\times\Omega$ 上积分并相加，利用边界条件得到

$$\frac{1}{2}\int_0^{T_1}\frac{\mathrm{d}}{\mathrm{d}t}I(t)\mathrm{d}t+\int_0^{T_1}\int_{\Gamma_1}[|\psi_t|^2+|\phi_t|^2+|\omega_t|^2]\mathrm{d}\Gamma_1\mathrm{d}t=0.$$

因此

$$2\int_0^{T_1}\int_{\Gamma_1}[|\psi_t|^2+|\phi_t|^2+|\omega_t|^2]\mathrm{d}\Gamma_1\mathrm{d}t=I(0)-I(T_1). \tag{5.13}$$

此外系统 (5.12) 的弱解分解为 $\Phi=\widetilde{\Phi}+\widehat{\Phi}$, 其中， $\widetilde{\Phi}$ 是系统 (5.3) 的弱解， $\widehat{\Phi}=(\widehat{\psi},\widehat{\phi},\widehat{\omega})\in C^0([0,T];W)\cap C^1([0,T];H)$ 是以下系统弱解

$$\begin{cases}\rho_1\widehat{\psi}_{tt}-D(\widehat{\psi}_{xx}+\dfrac{1-\mu}{2}\widehat{\psi}_{yy}+\dfrac{1+\mu}{2}\widehat{\phi}_{xy})+K(\widehat{\psi}+\widehat{\omega}_x)=0, \quad (x,y,t)\in\Omega\times(0,T),\\ \rho_1\widehat{\phi}_{tt}-D(\widehat{\phi}_{yy}+\dfrac{1-\mu}{2}\widehat{\phi}_{xx}+\dfrac{1+\mu}{2}\widehat{\psi}_{xy})+K(\widehat{\phi}+\widehat{\omega}_y)=0, \quad (x,y,t)\in\Omega\times(0,T),\\ \rho_2\widehat{\omega}_{tt}-K[(\widehat{\psi}+\widehat{\omega}_x)_x+(\widehat{\phi}+\widehat{\omega}_y)_y]=0, \quad (x,y,t)\in\Omega\times(0,T),\\ \widehat{\psi}=\widehat{\phi}=\widehat{\omega}=0, \quad (x,y,t)\in\Gamma_0\times(0,T),\\ D[\nu_1\widehat{\psi}_x+\mu\nu_1\widehat{\phi}_y+\dfrac{1-\mu}{2}(\widehat{\psi}_y+\widehat{\phi}_x)\nu_2]=-\psi_t, \quad (x,y,t)\in\Gamma_1\times(0,T),\\ D[\nu_2\widehat{\phi}_y+\mu\nu_2\widehat{\psi}_x+\dfrac{1-\mu}{2}(\widehat{\psi}_y+\widehat{\phi}_x)\nu_1]=-\phi_t, \quad (x,y,t)\in\Gamma_1\times(0,T),\\ K(\dfrac{\partial\widehat{\omega}}{\partial\nu}+\nu_1\widehat{\psi}+\nu_2\widehat{\phi})=-\omega_t, \quad (x,y,t)\in\Gamma_1\times(0,T),\\ (\widehat{\psi}(x,y,0),\widehat{\phi}(x,y,0),\widehat{\omega}(x,y,0))=(0,0,0), \quad (x,y)\in\Omega,\\ (\widehat{\psi}_t(x,y,0),\widehat{\phi}_t(x,y,0),\widehat{\omega}_t(x,y,0))=(0,0,0), \quad (x,y)\in\Omega.\end{cases} \tag{5.14}$$

由能观性条件及命题 2.1 中 (2.4) 式知

$$\begin{aligned}I(0)=\|(\Phi_{01}\Phi_{02})\|_{\mathcal{H}}^2&\le\frac{1}{C}\int_0^{T_1}\int_{\Gamma_1}(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t\\ &\le\frac{1}{C}\left[\int_0^{T_1}\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t+\int_0^{T_1}\int_{\Gamma_1}(|\widehat{\psi}_t|^2+|\widehat{\phi}_t|^2+|\widehat{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t\right]\\ &\le C'\int_0^{T_1}\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t.\end{aligned} \tag{5.15}$$

由 (5.13) 和 (5.15) 式得

$$I(T_1)-I(0)=-2\int_0^{T_1}\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t\le-\frac{2}{C'}I(0)\le-\frac{2}{C'}I(T_1),$$

取 $\alpha=\frac{\ln(1+\frac{1}{C'})}{T_1}$, 则有

$$I(T_1)\le\mathrm{e}^{-\alpha T_1}I(0). \tag{5.16}$$

因此，对于 $\forall k\in\mathbb{N}^+$, 有 $I(kT_1)\le\mathrm{e}^{-\alpha T_1}I((k-1)T_1)$. 于是对于 $\forall t\in[0,\infty)$, 存在 $k\in\mathbb{N}^+$, 使得 $t\in[kT_1,(k+1)T_1)$ 有

$$I(t)\le I(kT_1)\le\mathrm{e}^{-\alpha kT_1}I(0)=(1+\frac{2}{C'})\mathrm{e}^{-\alpha(k+1)T_1}I(0)\le(1+\frac{2}{C'})\mathrm{e}^{-\alpha t}I(0).$$

记 $M=1+\frac{2}{C'}$, 则 $I(t)\le M\mathrm{e}^{-\alpha t}I(0)$, 即 $\|(\Phi,\Phi_t)\|_{\mathcal{H}}^2\le M\mathrm{e}^{-\alpha t}\|(\Phi_{01},\Phi_{02})\|_{\mathcal{H}}^2$.

(II) 必要性

对于 $\forall t>0$, 将 ψ_t,ϕ_t,ω_t 分别乘以系统 (5.14) 前三式，在 $[0,t]\times\Omega$ 上积分并相加，分部积分得

$$2\int_0^t\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t=I(0)-I(t).$$

又由于系统是指数稳定的，故存在足够大的 $T'>0$, 使得

$$\int_0^{T'}\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t\ge\frac{1}{4}I(0). \tag{5.17}$$

再用 $\widehat{\psi}_t,\widehat{\phi}_t,\widehat{\omega}_t$ 分别乘以系统 (5.14) 前三个等式，在 $[0,T']\times\Omega$ 上积分并相加得

$$\begin{aligned}&\int_0^{T'}\int_\Omega[\rho_1(\widehat{\psi}_{tt}\widehat{\psi}_t+\widehat{\phi}_{tt}\widehat{\phi}_t)+\rho_2\widehat{\omega}_{tt}\widehat{\omega}_t]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\&=\int_0^{T'}\int_\Omega[L_1\{\widehat{\psi},\widehat{\phi},\widehat{\omega}\}\widehat{\psi}_t+L_2\{\widehat{\psi},\widehat{\phi},\widehat{\omega}\}\widehat{\phi}_t+L_3\{\widehat{\psi},\widehat{\phi},\widehat{\omega}\}\widehat{\omega}_t]\mathrm{d}x\mathrm{d}y\mathrm{d}t,\end{aligned}$$

分部积分得

$$\begin{aligned}&\int_\Omega[\rho_1(|\widehat{\psi}_t(T')|^2+|\widehat{\phi}_t(T')|^2)+\rho_2|\widehat{\omega}_t(T')|^2]\mathrm{d}x\mathrm{d}y\\&+a(\widehat{\psi}(T'),\widehat{\phi}(T'),\widehat{\omega}(T');\widehat{\psi}(T'),\widehat{\phi}(T'),\widehat{\omega}(T'))+2\int_0^{T'}\int_{\Gamma_1}(\widehat{\psi}_t\psi_t+\widehat{\phi}_t\phi_t+\widehat{\omega}_t\omega_t)\mathrm{d}\Gamma_1\mathrm{d}t=0,\end{aligned}$$

亦即

$$0\le\frac{1}{2}\|(\widehat{\Phi}(T'),\widehat{\Phi}_t(T'))\|_{\mathcal{H}}^2=-\int_0^{T'}\int_{\Gamma_1}(\widehat{\psi}_t(\widehat{\psi}_t+\widetilde{\psi}_t)+\widehat{\phi}_t(\widehat{\phi}_t+\widetilde{\phi}_t)+\widehat{\omega}_t(\widehat{\omega}_t+\widetilde{\omega}_t))\mathrm{d}\Gamma_1\mathrm{d}t. \tag{5.18}$$

因此，由 Cauchy-Schwarz 以及 Young 不等式，我们有

$$\int_0^{T'}\int_{\Gamma_1}(|\widehat{\psi}_t|^2+|\widehat{\phi}_t|^2+|\widehat{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t\le C''\int_0^{T'}\int_{\Gamma_1}(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t. \tag{5.19}$$

注意到： $\psi=\widehat{\psi}+\widetilde{\psi},\phi=\widehat{\phi}+\widetilde{\phi},\omega=\widehat{\omega}+\widetilde{\omega}$. 于是，我们有

$$\begin{aligned}&\int_0^{T'}\int_{\Gamma_1}(|\widehat{\psi}_t|^2+|\widehat{\phi}_t|^2+|\widehat{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t+\int_0^{T'}\int_{\Gamma_1}(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t\\&\ge\frac{1}{2}\int_0^{T'}\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t.\end{aligned} \tag{5.20}$$

上式结合 (5.17) 和 (5.19) 式得

$$\begin{aligned}\|(\Phi_{01},\Phi_{02})\|_{\mathcal{H}}^2=I(0)&\le4\int_0^{T'}\int_{\Gamma_1}(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t\\&\le8\Big[\int_0^{T'}\int_{\Gamma_1}(|\widehat{\psi}_t|^2+|\widehat{\phi}_t|^2+|\widehat{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t+\int_0^{T'}\int_{\Gamma_1}(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t\Big]\\&\le8(1+C'')\int_0^{T'}\int_{\Gamma_1}(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}\Gamma_1\mathrm{d}t.\end{aligned}$$

故能观性条件 (5.1) 得证．证毕. ▮

6 次最优性和最优轨线指数稳定性

接下来我们研究系统 (2.2) 的次最优性与最优轨线的指数稳定性.

命题 6.1 对于任意初始对 $\mathcal{Y}_0=(\Phi_{01},\Phi_{02})\in\mathcal{H}$ 及任意 $T>0$, 系统 (2.2) 存在控制 $\widehat{U}=(\widehat{u}_1,\widehat{u}_2,\widehat{u}_3)\in L^2([0,T];\mathcal{U})$ 使得

$$V_T(\mathcal{Y}_0)\le J_T(\mathcal{Y}_0;\widehat{U})\le\gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}},\tag{6.1}$$

其中 $\gamma_1(\cdot)$ 是一个连续、不减的有界函数. 进一步，存在只与 T 有关的正常数 $\gamma_2(T)$, 使得

$$V_T(\mathcal{Y}_0)\ge\gamma_2(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}}.\tag{6.2}$$

证 取 $u_1=-\psi_t,u_2=-\phi_t,u_3=-\omega_t$, 那么

$$I(t)\le M\mathrm{e}^{-\alpha t}I(0),\quad\forall t\in[0,T].\tag{6.3}$$

对 (6.3) 式两边在 $[0,T]$ 上积分得

$$\int_0^T I(t)\mathrm{d}t\le\frac{M}{\alpha}(1-\mathrm{e}^{-\alpha T})I(0),$$

再由 (5.13) 式得

$$\int_0^T\int_{\Gamma_1}[|\psi_t|^2+|\phi_t|^2+|\omega_t|^2]\mathrm{d}\Gamma_1\mathrm{d}t\le\frac{1}{2}I(0).$$

取 $\gamma_1(T)=\frac{M}{2\alpha}(1-\mathrm{e}^{-\alpha T})+\frac{\beta}{4}$, 由值函数的定义，我们有

$$\begin{aligned}V_T(\mathcal{Y}_0)&\le\frac{1}{2}\int_0^T I(t)\mathrm{d}t+\frac{\beta}{2}\int_0^T\int_{\Gamma_1}[|\psi_t|^2+|\phi_t|^2+|\omega_t|^2]\mathrm{d}\Gamma_1\mathrm{d}t\\&\le[\frac{M}{2\alpha}(1-\mathrm{e}^{-\alpha T})+\frac{\beta}{4}]I(0)=\gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}},\end{aligned}$$

即 (6.1) 式成立.

为了证明 (6.2) 式，对于任意 $U=(u_1,u_2,u_3)\in L^2([0,T];\mathcal{U})$, 我们应用叠加原理，将系统 (2.2) 的弱解表示为 $\Phi=\widetilde{\Phi}+\widehat{\Phi}$, 其中 $\widetilde{\Phi}=(\widetilde{\psi},\widetilde{\phi},\widetilde{\omega})$ 是系统 (5.3) 的弱解， $\widehat{\Phi}=(\widehat{\psi},\widehat{\phi},\widehat{\omega})$ 是下面系统的弱解

$$\begin{cases}\rho_1\widehat{\psi}_{tt}-D(\widehat{\psi}_{xx}+\dfrac{1-\mu}{2}\widehat{\psi}_{yy}+\dfrac{1+\mu}{2}\widehat{\phi}_{xy})+K(\widehat{\psi}+\widehat{\omega}_x)=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \rho_1\widehat{\phi}_{tt}-D(\widehat{\phi}_{yy}+\dfrac{1-\mu}{2}\widehat{\phi}_{xx}+\dfrac{1+\mu}{2}\widehat{\psi}_{xy})+K(\widehat{\phi}+\widehat{\omega}_y)=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \rho_2\widehat{\omega}_{tt}-K[(\widehat{\psi}+\widehat{\omega}_x)_x+(\widehat{\phi}+\widehat{\omega}_y)_y]=0,\quad(x,y,t)\in\Omega\times(0,T),\\ \widehat{\psi}=\widehat{\phi}=\widehat{\omega}=0,\quad(x,y,t)\in\Gamma_0\times(0,T),\\ D[\nu_1\widehat{\psi}_x+\mu\nu_1\widehat{\phi}_y+\dfrac{1-\mu}{2}(\widehat{\psi}_y+\widehat{\phi}_x)\nu_2]=u_1,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ D[\nu_2\widehat{\phi}_y+\mu\nu_2\widehat{\psi}_x+\dfrac{1-\mu}{2}(\widehat{\psi}_y+\widehat{\phi}_x)\nu_1]=u_2,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ K(\dfrac{\partial\widehat{\omega}}{\partial\nu}+\nu_1\widehat{\psi}+\nu_2\widehat{\phi})=u_3,\quad(x,y,t)\in\Gamma_1\times(0,T),\\ (\widehat{\psi}(x,y,0),\widehat{\phi}(x,y,0),\widehat{\omega}(x,y,0))=(0,0,0),\quad(x,y)\in\Omega,\\ (\widehat{\psi}_t(x,y,0),\widehat{\phi}_t(x,y,0),\widehat{\omega}_t(x,y,0))=(0,0,0),\quad(x,y)\in\Omega.\end{cases}\tag{6.4}$$

由能观性条件 (5.2) 和 (2.4) 得

$$
\begin{aligned}
\|(\Phi_{01},\Phi_{02})\|_{\mathcal{H}}^2 &\le \frac{1}{C_2}\int_0^T\int_\Omega(|\widetilde{\psi}_t|^2+|\widetilde{\phi}_t|^2+|\widetilde{\omega}_t|^2)\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\le \frac{1}{C_2}\int_0^T\left[\int_\Omega(|\psi_t|^2+|\phi_t|^2+|\omega_t|^2)\mathrm{d}x\mathrm{d}y+\int_\Omega(|\widehat{\psi}_t|^2+|\widehat{\phi}_t|^2+|\widehat{\omega}_t|^2)\mathrm{d}x\mathrm{d}y\right]\mathrm{d}t\\
&\le C_3\bigg\{\int_0^T\int_\Omega[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega[\rho_1(|\widehat{\psi}_t|^2+|\widehat{\phi}_t|^2)+\rho_2|\widehat{\omega}_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\bigg\}\\
&\le C_3\bigg\{\int_0^T\int_\Omega[\rho_1(|\psi_t|^2+|\phi_t|^2)+\rho_2|\omega_t|^2]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\quad+TC^2\int_0^T\int_{\Gamma_1}(|u_1|^2+|u_2|^2)+|u_3|^2]\mathrm{d}\Gamma_1\mathrm{d}t\bigg\}+C_3\int_0^T a(\psi,\phi,\omega;\psi,\phi,\omega)\mathrm{d}t\\
&\le C''(T)\int_0^T\left[\frac{1}{2}\|\mathcal{Y}\|_{\mathcal{H}}^2+\frac{\beta}{2}\|U\|_{\mathcal{U}}^2\right]\mathrm{d}t\\
&= C''(T)\int_0^T\ell(\mathcal{Y}(t),U(t))\mathrm{d}t.
\end{aligned}
$$

由控制变量 $U=(u_1,u_2,u_3)\in L^2([0,T];\mathcal{U})$ 的任意性, 我们得到只与 T 有关的正常数 $\gamma_2(T)=\frac{1}{C''(T)}$ 使得 (6.2) 式成立. 证毕. ▍

引理 6.1 对于任意初始值 $\mathcal{Y}_0\in\mathcal{H}$ 以及 $\delta>0$, 当 $T>\delta$ 时, 则, 对所有的 $\widehat{t}\in[\delta,T]$, 有

$$
V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}_0,0))\le\int_\delta^{\widehat{t}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),\mathcal{U}_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+\gamma_1(T+\delta-\widehat{t})\left\|\mathcal{Y}_T^*(\widehat{t},\mathcal{Y}_0,0)\right\|_{\mathcal{H}}^2 \tag{6.5}
$$

和对所有的 $\widetilde{t}\in[0,T]$, 有

$$
\int_{\widetilde{t}}^T\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),\mathcal{U}_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\le\gamma_1(T-\widetilde{t})\left\|\mathcal{Y}_T^*(\widetilde{t},\mathcal{Y}_0,0)\right\|_{\mathcal{H}}^2. \tag{6.6}
$$

证 取 $\xi=\frac{1}{2}\min\{1,\beta\}$, 对于任意 $\mathcal{Y}_0\in\mathcal{H}$ 及 $\bar{t}\in[0,T]$, 由 (1.4) 式得

$$
\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\ge\xi[\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2+\|U_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{U}}^2]. \tag{6.7}
$$

因此，我们有

$$
\begin{aligned}
\xi\int_0^{\bar{t}}[\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2+\|U_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{U}}^2]\mathrm{d}t &\le\int_0^{\bar{t}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\\
&=V_T(\mathcal{Y}_0)-V_{T-\bar{t}}(\mathcal{Y}_T^*(\bar{t},\mathcal{Y}_0,0)).
\end{aligned}
$$

又由 (2.4) 式得

$$
\begin{aligned}
\|\mathcal{Y}_T^*(\bar{t},\mathcal{Y}_0,0)\|_{\mathcal{H}}^2 &\le C'[\|\mathcal{Y}_0\|_{\mathcal{H}}^2+\int_0^{\bar{t}}[\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2+\|U_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{U}}^2]\mathrm{d}t]\\
&\le C'[\|\mathcal{Y}_0\|_{\mathcal{H}}^2+\frac{1}{\xi}(V_T(\mathcal{Y}_0)-V_{T-\bar{t}}(\mathcal{Y}_T^*(\bar{t},\mathcal{Y}_0,0)))]\\
&\le C'[\|\mathcal{Y}_0\|_{\mathcal{H}}^2+\frac{1}{\xi}V_T(\mathcal{Y}_0)]\le C'(1+\frac{\gamma_1(T)}{\xi})\|\mathcal{Y}_0\|_{\mathcal{H}}^2.
\end{aligned}
$$

因此，对于任意 $\bar{t}\in[0,T]$, 都有 $\mathcal{Y}_T^*(\bar{t},\mathcal{Y}_0,0)\in\mathcal{H}$.

对于任意 $\widehat{t}\in[\delta,T]$, 我们有

$$\begin{aligned}V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}_0,0)) &= V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}^*(\delta),\delta))\\ &= \int_\delta^{T+\delta}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}^*(\delta),\delta),U_T^*(t,\mathcal{Y}^*(\delta),\delta))\mathrm{d}t\\ &= \int_\delta^{\widehat{t}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}^*(\delta),\delta),U_T^*(t,\mathcal{Y}^*(\delta),\delta))\mathrm{d}t+V_{T+\delta-\widehat{t}}(\mathcal{Y}_T^*(\widehat{t},\mathcal{Y}^*(\delta),\delta)).\end{aligned}$$

又 $\mathcal{Y}_T^*(\cdot,\mathcal{Y}^*(\delta),\delta)$ 是系统在区间 $[\delta,T+\delta]$ 的最优解，根据贝尔曼最优性原理，我们有

$$\begin{aligned}V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}_0,0)) &\le \int_\delta^{\widehat{t}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+V_{T+\delta-\widehat{t}}(\mathcal{Y}_T^*(\widehat{t},\mathcal{Y}_0,0))\\ &\le \int_\delta^{\widehat{t}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+\gamma_1(T+\delta-\widehat{t})\|\mathcal{Y}_T^*(\widehat{t},\mathcal{Y}_0,0)\|_{\mathcal{H}}^2.\end{aligned}$$

因此， (6.5) 式成立.

接下来证明 (6.6) 式. 对于任意 $\widetilde{t}\in[0,T]$, 我们有

$$\begin{aligned}V_T(\mathcal{Y}_0) &= \int_0^{\widetilde{t}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+\int_{\widetilde{t}}^T\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\\ &= \int_0^{\widetilde{t}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+V_{T-\widetilde{t}}(\mathcal{Y}_T^*(\widetilde{t},\mathcal{Y}_0,0))\\ &\le \int_0^{\widetilde{t}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+\gamma_1(T-\widetilde{t})\|\mathcal{Y}_T^*(\widetilde{t},\mathcal{Y}_0,0)\|_{\mathcal{H}}^2.\end{aligned}$$

因此，对所有 $\widetilde{t}\in[0,T]$ 都有

$$\int_{\widetilde{t}}^T\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),\mathcal{U}_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\le\gamma_1(T-\widetilde{t})\left\|\mathcal{Y}_T^*(\widetilde{t},\mathcal{Y}_0,0)\right\|_{\mathcal{H}}^2.$$

证毕. ∎

定理 6.1 对于任意初始值 $\mathcal{Y}_0\in\mathcal{H}$, 以及样本时间 $\delta>0$ 和预测时间 $T>\delta$, 存在只与 T 有关的函数

$$\sigma_1(T)=1+\frac{\gamma_1(T)}{\xi(T-\delta)},\quad \sigma_2=\frac{\gamma_1(T)}{\xi\delta},$$

使得以下估计式

$$V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}_0,0))\le\sigma_1(T)\int_\delta^T\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t \tag{6.8}$$

和

$$\int_\delta^T\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\le\sigma_2(T)\int_0^\delta\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t \tag{6.9}$$

成立.

证 由于 $\mathcal{Y}_T^*(\cdot,\mathcal{Y}_0,0)\in C([0,T];\mathcal{H})$, 故一定存在 $t_1\in[\delta,T],t_2\in[0,\delta]$, 使得

$$t_1=\arg\min_{t\in[\delta,T]}\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2,\quad t_2=\arg\min_{t\in[0,\delta]}\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2.$$

由 (6.5) 和 (6.7) 式可得

$$\begin{aligned}V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}_0,0)) &\leq \int_\delta^{t_1}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+\gamma_1(T+\delta-t_1)\|\mathcal{Y}_T^*(t_1,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2\\&\leq \int_\delta^{T}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+\gamma_1(T)\|\mathcal{Y}_T^*(t_1,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2\\&\leq \int_\delta^{T}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t+\frac{\gamma_1(T)}{T-\delta}\int_\delta^T\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2\mathrm{d}t\\&\leq (1+\frac{\gamma_1(T)}{\xi(T-\delta)})\int_\delta^{T}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t.\end{aligned}$$

即 (6.8) 成立.

同理，由 (6.6) 和 (6.7) 式得

$$\begin{aligned}\int_\delta^{T}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t &\leq \int_{t_2}^{T}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\\&\leq \gamma_1(T-t_2)\|\mathcal{Y}_T^*(t_2,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2\\&\leq \frac{\gamma_1(T)}{\delta}\int_0^\delta\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2\mathrm{d}t\\&\leq \frac{\gamma_1(T)}{\xi\delta}\int_0^\delta\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t.\end{aligned}$$

因此 (6.9) 式成立．证毕. ▮

定理 6.2 对于任意初始值 $\mathcal{Y}_0\in\mathcal{H}$ 以及样本时间 $\delta>0$, 存在 $\widetilde{T}>\delta$ 和 $\kappa\in(0,1)$, 当 $T\geq\widetilde{T}$ 时，使得

$$V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}_0,0))\leq V_T(\mathcal{Y}_0)-\kappa\int_0^\delta\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t. \tag{6.10}$$

成立.

证 由定理 6.1 得

$$\begin{aligned}&V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}_0,0))-V_T(\mathcal{Y}_0)\\&=V_T(\mathcal{Y}_T^*(\delta,\mathcal{Y}_0,0))-\int_0^T\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\\&\leq(\sigma_1(T)-1)\int_\delta^T\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t-\int_0^\delta\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\\&\leq[\sigma_2(T)(\sigma_1(T)-1)-1]\int_0^\delta\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t.\end{aligned}$$

于是，当 $T\to\infty$ 时， $1-\sigma_2(T)(\sigma_1(T)-1)=1-\frac{\gamma_1^2(T)}{\xi^2\delta(T-\delta)}\to1$. 因此，存在 $\widetilde{T}>\delta$, 使得当 $T\geq\widetilde{T}$ 时，存在 $\kappa\in(0,1)$, 使得 $1-\sigma_2(T)(\sigma_1(T)-1)>\kappa$. 因此 (6.10) 式成立．证毕. ▮

定理 6.3 对于给定的样本时间 $\delta>0$, 则存在 $\widetilde{T}>\delta$ 和 $\kappa\in(0,1)$, 以及对每一预测时间 $T\geq\widetilde{T}$ 以及初始值 $\mathcal{Y}_0\in\mathcal{H}$, 滚动时域控制 $U_T^*(\cdot)$ 使得

$$\kappa V_\infty(\mathcal{Y}_0)\leq\kappa J_\infty(\mathcal{Y}_0,U_T^*(\cdot))\leq V_T(\mathcal{Y}_0)\leq V_\infty(\mathcal{Y}_0) \tag{6.11}$$

成立. 进一步，我们可以得到最优轨线是指数稳定的，即存在只与 δ,κ 有关的正数 M,η 使得

$$\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|\leq M\mathrm{e}^{-\eta t}. \tag{6.12}$$

成立.

证 由于 $\kappa V_\infty(\mathcal{Y}_0)\leq\kappa J_\infty(\mathcal{Y}_0,U_T^*(\cdot))$ 和 $V_T(\mathcal{Y}_0)\leq V_\infty(\mathcal{Y}_0)$ 显然成立. 因此，我们只需证明 $\kappa J_\infty(\mathcal{Y}_0,U_T^*(\cdot))\leq V_T(\mathcal{Y}_0)$ 即可.

对于给定 $\delta>0$, 取 $t_k=k\delta$, 由定理 6.2 可得，存在 $\widetilde{T}>\delta$ 和 $\kappa\in(0,1)$, 当 $T\geq\widetilde{T}$ 时，使得

$$\begin{aligned}&V_T(\mathcal{Y}_T^*(t_{k+1},\mathcal{Y}(t_k),t_k))\\&\leq V_T(\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_k),t_k))-\kappa\int_{t_k}^{t_{k+1}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}(t_k),t_k),U_T^*(t,\mathcal{Y}(t_k),t_k))\mathrm{d}t.\end{aligned} \tag{6.13}$$

因此，由递推得

$$V_T(\mathcal{Y}_T^*(t_{k+1},\mathcal{Y}(t_k),t_k))\leq V_T(\mathcal{Y}_0)-\kappa\int_0^{t_{k+1}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t,k=0,1,2,\cdots. \tag{6.14}$$

上式蕴含着

$$\kappa\int_0^{t_{k+1}}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\leq V_T(\mathcal{Y}_0). \tag{6.15}$$

令 $k\to\infty$, 则有

$$\kappa J_\infty(\mathcal{Y}_0,U_T^*(\cdot))=\kappa\int_0^\infty\ell(\mathcal{Y}_T^*(t,\mathcal{Y}_0,0),U_T^*(t,\mathcal{Y}_0,0))\mathrm{d}t\leq V_T(\mathcal{Y}_0),$$

从而 (6.11) 式成立.

对于任意 $k\in\mathbb{N}^+$, 由 (6.6) 式得

$$\begin{aligned}&V_T(\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_{k-1}),t_{k-1}))-V_T(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1}))\\&\leq-\kappa\int_{t_{k-1}}^{t_k}\ell(\mathcal{Y}_T^*(t,\mathcal{Y}(t_{k-1}),t_{k-1}),U_T^*(t,\mathcal{Y}(t_{k-1}),t_{k-1}))\mathrm{d}t\\&\leq-\kappa V_\delta(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1})).\end{aligned} \tag{6.16}$$

又由 (6.1) 和 (6.2) 式知，对于任意 $\mathcal{Y}_0\in\mathcal{H}$, 我们有

$$V_\delta(\mathcal{Y}_0)\geq\gamma_2(\delta)\|\mathcal{Y}_0\|_{\mathcal{H}}^2\geq\frac{\gamma_2(\delta)}{\gamma_1(T)}V_T(\mathcal{Y}_0).$$

因此

$$V_\delta(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1}))\geq\frac{\gamma_2(\delta)}{\gamma_1(T)}V_T(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1})). \tag{6.17}$$

再由 (6.16) 和 (6.17) 式得

$$\begin{aligned}&V_T(\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_{k-1}),t_{k-1}))-V_T(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1}))\\&\leq-\frac{\kappa\gamma_2(\delta)}{\gamma_1(T)}V_T(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1})),\end{aligned}$$

即

$$V_T(\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_{k-1}),t_{k-1}))\leq(1-\frac{\kappa\gamma_2(\delta)}{\gamma_1(T)})V_T(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1})).$$

记 $\mu=1-\frac{\kappa\gamma_2(\delta)}{\gamma_1(T)}$, 注意到 $0<\gamma_2(\delta)\leq\gamma_1(\delta)\leq\gamma_1(T)$ 且 $\kappa\in(0,1)$, 所以, $\mu\in(0,1)$, 且对 $k\in\mathbb{N}^+$ 有

$$V_T(\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_{k-1}),t_{k-1}))\leq\mu V_T(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1})).\tag{6.18}$$

若再记 $\eta=\frac{\ln|\mu|}{\delta}$, 则对 $k\in\mathbb{N}^+$ 有

$$\begin{aligned}\gamma_2(T)\|\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_{k-1}),t_{k-1})\|_{\mathcal{H}}^2&\leq V_T(\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_{k-1}),t_{k-1}))\\&\leq\mu V_T(\mathcal{Y}_T^*(t_{k-1},\mathcal{Y}(t_{k-1}),t_{k-1}))\\&\leq\mu^2V_T(\mathcal{Y}_T^*(t_{k-2},\mathcal{Y}(t_{k-2}),t_{k-2}))\leq\cdots\\&\leq\mu^kV_T(\mathcal{Y}_0)=\mathrm{e}^{-\eta t_k}V_T(\mathcal{Y}_0)\leq\mathrm{e}^{-\eta t_k}\gamma_1(T)\|\mathcal{Y}_0\|_{\mathcal{H}}^2.\end{aligned}$$

记 $\gamma=\frac{\gamma_1(T)}{\gamma_2(T)}$, 则对 $k\in\mathbb{N}^+$ 有

$$\|\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_{k-1}),t_{k-1})\|_{\mathcal{H}}^2\leq\gamma\mathrm{e}^{-\eta t_k}\|\mathcal{Y}_0\|_{\mathcal{H}}^2.\tag{6.19}$$

故对任意 $t>0$, 存在 $k\in\mathbb{N}^+$, 使得 $t\in[k\delta,(k+1)\delta]=[t_k,t_{k+1}]$, 且由 (2.4), (6.1) 和 (6.19) 式得

$$\begin{aligned}\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2&\leq C(\|\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_k),t_k)\|_{\mathcal{H}}^2+\int_{t_k}^{t_k+T}\|U_T^*(t,\mathcal{Y}(t_k),t_k)\|_{\mathcal{U}}^2\mathrm{d}t)\\&\leq C(\|\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_k),t_k)\|_{\mathcal{H}}^2+\frac{2}{\beta}V_T(\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_k),t_k)))\\&\leq C(\|\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_k),t_k)\|_{\mathcal{H}}^2+\frac{2\gamma_1(T)}{\beta}\|\mathcal{Y}_T^*(t_k,\mathcal{Y}(t_k),t_k)\|_{\mathcal{H}}^2)\\&\leq C(1+\frac{2\gamma_1(T)}{\beta})\gamma\mathrm{e}^{-\eta t_k}\|\mathcal{Y}_0\|_{\mathcal{H}}^2\\&=\gamma C(1+\frac{2\gamma_1(T)}{\beta})\frac{1}{1-\frac{\kappa\gamma_2(\delta)}{\gamma_1(T)}}\mathrm{e}^{-\eta t_{k+1}}\|\mathcal{Y}_0\|_{\mathcal{H}}^2\\&\leq\gamma C(1+\frac{2\gamma_1(T)}{\beta})\frac{1}{1-\frac{\kappa\gamma_2(\delta)}{\gamma_1(T)}}\mathrm{e}^{-\eta t}\|\mathcal{Y}_0\|_{\mathcal{H}}^2.\end{aligned}$$

取 $M=\gamma C(1+\frac{2\gamma_1(T)}{\beta})\frac{1}{1-\frac{\kappa\gamma_2(\delta)}{\gamma_1(T)}}\|\mathcal{Y}_0\|_{\mathcal{H}}^2$, 故有 $\|\mathcal{Y}_T^*(t,\mathcal{Y}_0,0)\|_{\mathcal{H}}^2\leq M\mathrm{e}^{-\eta t}$. 证毕. ▮

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Stability and Optimality of 2-D Mindlin-Timoshenko Plate System

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Abstract: In this paper, 2-D Mindlin Timoshenko plate system with local boundary control is studied. By using the receding horizon control method, the infinite time domain optimality problem is transformed into the finite time domain optimality problem. With the help of the multiplier technique, a priori estimation is made for any finite time domain system, and the observability inequality is obtained, which proves that the energy of the system is uniformly exponentially decay. Furthermore, with the aid of dual system, by means of the variational principle and Bellman optimality principle, the suboptimal conditions of the system in infinite time domain are obtained, and it is proved that the optimal trajectory is also exponential decay.

Key words: 2-D Mindlin Timoshenko plate; Receding horizon control method; Optimality; Exponential decay.

MR(2010) Subject Classification: 35B37; 93B52