数学物理学报
Acta Mathematica Scientia
2021,**41A**(4):1218–1234
http://actams.wipm.ac.cn

转换机制下具有非线性扰动的随机 SIVS 传染病模型的定性分析

张仲华* 张倩
(西安科技大学理学院 西安 710054)

摘要：该文考虑随机环境因素的影响，建立了一类转换机制下具有非线性扰动的 SIVS 模型．对于具有白噪声的非自治随机 SIVS 流行病系统，给出了解的随机有界性和随机持久性的结果，并利用李雅普诺夫函数和 Has'minskii 周期解理论证明了非平凡正周期解的存在性．对于具有马尔科夫变换的系统，建立了遍历平稳分布的充分条件，分别得到了染病者在平均意义下持久性的阈值和灭绝性的阈值．最后，通过数值模拟支撑了理论结果．

关键词：随机 SIVS 传染病模型；非线性扰动；马尔科夫链；非线性发病率．

MR(2010) 主题分类：60H30 **中图分类号**：O211.63 **文献标识码**：A

文章编号：1003-3998(2021)04-1218-17

1 引言

生态环境的不断破坏和国际交流的日趋频繁是影响传染病的暴发和流行的重要因素．近年来，新冠肺炎和非典型肺炎等新型传染病不断暴发以及肝炎、肺结核等原有疾病的反复发作给人类的健康和社会经济的发展带来了严重威胁．运用数学模型研究传染病的传播规律和流行趋势对疾病的预防和控制具有重要的理论和实践意义．

疫苗接种是控制和消除传染病最有效的方法之一，为了寻求疾病的最佳控制策略，越来越多的学者致力于具有预防接种的数学模型的动力学研究．早期的传染病动力学中关于预防接种的工作主要是建立确定性模型，并围绕平衡点的存在性、稳定性及多种分歧现象的存在性等问题进行研究，如文献 [1–8] 等．

然而，疾病传播过程中往往受到复杂环境中不确定因素的影响．具有不同形式噪声干扰的传染病模型近年来受到普遍关注，如文献 [9–12] 等．赵亚男等人 [9] 在文献 [3] 的基础上对感染系数进行随机扰动并提出了一类 SIVS 模型，研究发现当噪声较大时，不论基本再生数是否小于 1, 感染人群的数量都将指数衰减为 0. 刘群和蒋达清 [10] 采取了同样的扰动方法建立了一类接种后不完全免疫的 SIVS 模型，给出了疾病灭绝的条件，且发现较大的噪声能够有效地抑制疾病的流行．在文献 [11–12] 中，赵亚男和蒋达清假设随机扰动项与易感者类

收稿日期: 2020-09-11; 修订日期: 2021-03-12
E-mail: wwwzhangzhonghua@163.com;994926272@qq.com
基金项目：国家自然科学基金 (11201277)
Supported by the NSFC(11201277)
∗ 通讯作者

(S) 、感染者类 (I) 和接种者类 (V) 成比例建立了一类自治 SIVS 随机模型，研究了由确定型模型的基本再生数决定的遍历性等性质，并且分别给出了决定疾病灭绝及持久的阈值条件.

以上论文中随机干扰项均为系统状态的线性函数，在文献 [13] 中张伟伟等人使用白噪声模拟环境波动，提出了一类具有非线性扰动的随机非自治 SIRI 模型，给出了解的随机有界性、随机持久性和疾病持续存在的解析结果，且对于具有马尔科夫转换的系统得到了遍历平稳分布和正常返存在的充分条件.

事实上，影响疾病传播的环境噪声不仅有上述的白噪声 (一种功率谱密度为常数的随机过程，通常用布朗运动广义均方导数表示), 还有一种强度大但数量少的有色噪声，又称为电报信号噪声 [14–15]. 在现实生活中，很多疾病的传播受季节、气候、政治人文因素的改变影响较大，如 Covid-19, SARS, 流感等. 这一类因素的功率谱密度函数不为常数，不能单纯地用布朗运动来描述其随机性，因此一些学者在传染病建模过程中运用 Markov 切换描述其随机性，如文献 [16–17] 等. 本文拟运用 Markov 切换描述有色噪声，建立具有非线性随机扰动及预防接种策略的传染病模型，并通过李雅普诺夫函数、 Chebyshev's 不等式、 Has'minskii 理论、强大数定理等方法研究模型的持久性、灭绝性和遍历性等性质.

2 模型的建立

胡俊娜 [18] 等人提出了转换机制 (环境状态之间的转换通常是无记忆的，下一次切换停留时间服从指数分布) 下具有非线性发病率的随机 SIVS 传染病模型

$$\begin{cases} \mathrm{d}S = [(1-q_{r(t)})A_{r(t)} - \beta_{r(t)}f(S(t)g(I(t)) - (\mu_{r(t)} + p_{r(t)})S(t) + \gamma_{r(t)}I(t) \\ \qquad +\varepsilon_{r(t)}V(t)]\mathrm{d}t + \sigma_{1r(t)}S(t)\mathrm{d}B_1(t), \\ \mathrm{d}I = [\beta_{r(t)}f(S(t)g(I(t)) - (\mu_{r(t)} + \gamma_{r(t)} + \alpha_{r(t)})I(t)]\mathrm{d}t + \sigma_{2r(t)}I(t)\mathrm{d}B_2(t), \\ \mathrm{d}V = [q_{r(t)}A_{r(t)} + p_{r(t)}S(t) - (\mu_{r(t)} + \varepsilon_{r(t)})V(t)]\mathrm{d}t + \sigma_{3r(t)}V(t)\mathrm{d}B_3(t), \end{cases} \tag{2.1}$$

其中 $S(t), I(t), V(t)$ 分别为 t 时刻易感人群，感染人群及接种疫苗后获得免疫人群的数量，$q_{r(t)}$ 为新生儿的接种率， $A_{r(t)}$ 为人口总输入率， $\beta_{r(t)}$ 为疾病传播率， $\mu_{r(t)}$ 为自然死亡率，$p_{r(t)}$ 为易感者的接种免疫率， $\alpha_{r(t)}$ 为因疾病造成的死亡率， $\gamma_{r(t)}$ 为患者的康复率， $\varepsilon_{r(t)}$ 为康复者的免疫失去率， $r(t)$ 为完备概率空间 $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ 上的右连续马尔科夫链，它的状态空间为 $\hbar = \{1, 2, 3\cdots, N\}$. 定义马尔科夫链的生成元 $\Gamma = (\gamma_{ij})_{N\times N}$ 为

$$\mathbb{P}\{r(t+\triangle) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\triangle + \circ(\triangle), i \neq j, \\ 1 + \gamma_{ij}\triangle + \circ(\triangle), i = j, \end{cases}$$

其中 $\triangle > 0, \gamma_{ij} \geq 0 (i \neq j)$ 是从 i 到 j 的转移概率， $\gamma_{ii} = -\sum\limits_{i\neq j}\gamma_{ij}$. 作者通过对函数 f 和 g 做一般性的假设，讨论了模型 (1.1) 解的遍历平稳分布性质及灭绝性和平均持久性.

不难验证，当 $g(I(t)) = I^m$ 时， $\frac{g(I)}{I} = I^{m-1}$, 若 $m > 1$, 则 $\frac{g(I)}{I}$ 关于 I 单调非减，作者未对此情况进行讨论. 基于此，本文拟考虑具有转换机制的非线性发病率 $\beta(r(t))SI^m(m > 1)$ 及非线性随机扰动项

$$S(\sigma_{11}(r(t)) + \sigma_{12}(r(t))S)\mathrm{d}B_1(t), I(\sigma_{21}(r(t)) + \sigma_{22}(r(t))I)\mathrm{d}B_2(t),$$

$$V(\sigma_{31}(r(t)) + \sigma_{32}(r(t))V)\mathrm{d}B_3(t).$$

建立如下 SIVS 模型

$$
\begin{cases}
\mathrm{d}S = [(1-q(r(t)))A(r(t)) - \beta(r(t))SI^m - (\mu(r(t)) + p(r(t)))S \\
\qquad +\gamma(r(t))I + \varepsilon(r(t))V]\mathrm{d}t + S(\sigma_{11}(r(t)) + \sigma_{12}(r(t))S)\mathrm{d}B_1(t), \\
\mathrm{d}I = [\beta(r(t))SI^m - (\mu(r(t)) + \gamma(r(t)) + \alpha(r(t)))I]\mathrm{d}t \\
\qquad +I(\sigma_{21}(r(t)) + \sigma_{22}(r(t))I)\mathrm{d}B_2(t), \\
\mathrm{d}V = [q(r(t))A(r(t)) + p(r(t))S - (\mu(r(t)) + \varepsilon(r(t)))V]\mathrm{d}t \\
\qquad +V(\sigma_{31}(r(t)) + \sigma_{32}(r(t))V)\mathrm{d}B_3(t),
\end{cases} \tag{2.2}
$$

其中 $B_i(t)$ $(i=1,2,3)$ 为标准布朗运动， $\sigma_{ij}(i=1,2,3,j=1,2)$ 为白噪声强度，其余参数与模型 (2.1) 中的意义相同，且假设布朗运动 $B_i(t)$ 独立于马尔科夫链 $r(t)$, 进一步假设 $r(t)$ 不可约, 这表明马尔科夫链 $r(t)$ 有唯一的平稳分布 $\pi=\{\pi_1,\pi_2,\cdots,\pi_N\}$. 平稳分布在 $\sum\limits_{h=1}^{N}\pi_h=1$ 和 $\pi_h>0$ 的情况下由 $\pi\Gamma$ 确定，其中 $h\in\hbar$. 本文假设 $i\neq j,\gamma_{ij}>0$.

为了讨论方便，假设 $A(r(t)),\beta(r(t)),\mu(r(t)),\gamma(r(t))$ 为正常数，其余参数为非负常数，且 $q(r(t))<1$, 记 $\mathbb{R}_+^n=\{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n: x_i>0, i=1,2,\cdots,n\}$. 对于任意的向量 $g=\{g(1),g(2),\cdots g(N)\}$, 定义 $\hat{g}=\min\limits_{k\in\hbar}\{g(k)\}$ 和 $\check{g}=\max\limits_{k\in\hbar}\{g(k)\}$. 对 $[0,+\infty]$ 上的可积函数 f, 定义

$$
\langle f\rangle_t=\frac{1}{t}\int_0^t f(s)\mathrm{d}s, f^u=\sup_{t\in[0,\infty]}f(t), f^l=\inf_{t\in[0,\infty]}f(t).
$$

除了各种环境噪声外，在现实中，环境的周期变化也是影响传染病传播的一个重要因素，基于系统 (2.2), 忽略有色噪声，我们建立如下具有周期系数的非自治传染病模型

$$
\begin{cases}
\mathrm{d}S = [(1-q(t))A(t) - \beta(t)SI^m - (\mu(t)+p(t))S + \gamma(t)I + \varepsilon(t)V]\mathrm{d}t \\
\qquad +S(\sigma_{11}(t) + \sigma_{12}(t)S)\mathrm{d}B_1(t), \\
\mathrm{d}I = [\beta(t)SI^m - (\mu(t)+\gamma(t)+\alpha(t))I]\mathrm{d}t + I(\sigma_{21}(t)+\sigma_{22}(t)I)\mathrm{d}B_2(t), \\
\mathrm{d}V = [q(t)A(t) + p(t)S - (\mu(t)+\varepsilon(t))V]\mathrm{d}t + V(\sigma_{31}(t)+\sigma_{32}(t)V)\mathrm{d}B_3(t),
\end{cases} \tag{2.3}
$$

其中所有系数均为正 T - 周期连续函数.

为了讨论模型 (2.2)–(2.3) 的性质，我们引入如下结论．首先考虑随机微分方程

$$
\begin{cases}
\mathrm{d}x(t) = f(x(t),t)\mathrm{d}t + g(x(t),t)\mathrm{d}B(t), \\
x(0) = x_0.
\end{cases} \tag{2.4}
$$

引理 2.1[19] 假设系统 (2.4) 的系数在 t 上是 T - 周期的，且满足线性增长条件和在每一个区域 $D_l\times[0,\infty), l>0$ 满足 Lipschitz 条件，进一步假设存在一个函数 $V=V(t,x)$ 在 $\mathbb{R}_n\times[0,\infty)$ 上对 x 是二次连续可微的，对 t 是一次连续可微的，在 t 上是 T - 周期的且满足下列条件

i) $\inf\limits_{\|x\|>l}V(t,x)\to\infty$, 当 $l\to\infty$,

ii) $LV(t,x)\leq -1, \|x\|>l$,

则系统 (2.4) 存在 T - 周期解，微分算子 L 由下式定义

$$
L=V_t+V_xf(x,t)+\frac{1}{2}\mathrm{Trace}[g^T(t)V_{xx}g(x,t)],
$$

其中

$$V_t=\frac{\partial V}{\partial t},V_x=(\frac{\partial V}{\partial x_1},V_x=\frac{\partial V}{\partial x_2},\cdots,V_x=\frac{\partial V}{\partial x_N}),V_{xx}(\frac{\partial^2 V}{\partial x_i\partial x_j})_{n\times n}.$$

为了给出状态转换下微分方程的结果，定义如下方程

$$\begin{cases}\mathrm{d}X(t)=f(X(t),r(t))\mathrm{d}t+g(X(t),r(t))\mathrm{d}B(t),\\ \quad x(0)=x_0,\quad r(0)=r_0,\end{cases}\tag{2.5}$$

其中 $f:\mathbb{R}^n\times\hbar\to\mathbb{R}^n, g:\mathbb{R}^n\times\hbar\to\mathbb{R}^{n\times l}$. 设 $A(x,k)=g(x,k)g^T(x,k)=(A_{ij})_{n\times n}$, $V:\mathbb{R}\times\hbar\to\mathbb{R}^n$ 均为二阶连续可微函数. 定义算子

$$L=\sum_{i=1}^n f_i(x,k)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^n A_{ij}\frac{\partial^2}{\partial x_i x_j}+\sum_{i\neq k\in\hbar}\gamma_{ki}[V(x,i)-V(x,k)].$$

引理 2.2[20] 若满足以下条件

i) 对任意的 $i\neq j,\gamma_{ij}>0$,

ii) 对于任意的 $k\in\hbar, D(x,k)=(d_{ij}(x,k))$ 是对称的，且对于任意的 $x,\xi\in\mathbb{R}^n$ 满足 $\sigma|\xi|^2\leq\langle D(x,k)\xi,\xi\rangle\leq\sigma^{-1}|\xi|^2$, 其中常数 $\sigma\in[0,1]$,

iii) 存在闭包非空的开集 D, 对于任意的 $k\in\hbar$, 存在二阶连续可微的非负函数 $V(\cdot,k):D^c\to\mathbb{R}$, 当 $\alpha>0$ 时，对任意的 $(x,k)\in D^c\times\hbar$, 有 $LV(\cdot,k)\leq-\alpha$,

则系统 (2.5) 是遍历和正常返的，也就是说有唯一的平稳分布 $\mu(\cdot,\cdot)$, 对任意 Borel 可测函数 $f:\mathbb{R}\times\hbar\to\mathbb{R}^n$ 满足 $\sum\limits_{k=1}^N\int_{\mathbb{R}^n}|f(x,k)|\mu(dk,k)<+\infty$, 有

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s),r(s))\mathrm{d}s=\sum_{k=1}^N\int_{\mathbb{R}^n}|f(x,k)|\mu(dk,k)\right\}=1.$$

3 系统 (2.3) 的随机有界性、持久性以及非平凡正周期解的存在性

定义 1[13] 若对于任意的 $\delta\in(0,1)$ 存在正常数 $\varsigma=\varsigma(\delta)$, 使得对于任意初始值 $X(0)=X_0\in\mathbb{R}^3_+$, 系统 (2.3) 的解 $X(t)$ 具有以下性质

$$\liminf_{t\to\infty}\mathbb{P}[|X(t)|>\varsigma]<\delta,$$

则称 $X(t)$ 是随机最终有界的.

定义 2[21] 若对于任意的 $\delta\in(0,1)$ 存在一对正常数 $\varsigma=\varsigma(\delta),\chi=\chi(\delta)$, 使得对于任意初始值 $X(0)=X_0\in\mathbb{R}^3_+$, 系统 (2.3) 的解 $X(t)$ 具有以下性质

$$\liminf_{t\to\infty}\mathbb{P}[|X(t)|\leq\varsigma]\geq1-\delta,\qquad \liminf_{t\to\infty}\mathbb{P}[|X(t)|>\chi]\geq1-\delta,$$

则称 $X(t)$ 是随机持久的.

定义参数

$$\begin{aligned}\lambda_1&=\mu^l-(\sigma_{12}^{2u}\vee\sigma_{22}^{2u}\vee\sigma_{32}^{2u}),\\ C&=A^u+2(\sigma_{11}^u\sigma_{12}^u\vee\sigma_{21}^u\sigma_{22}^u\vee\sigma_{31}^u\sigma_{32}^u)+\frac{1}{4A^l}(\mu^l+\mu^u+\alpha^u+(\sigma_{11}^{2u}\vee\sigma_{21}^{2u}\vee\sigma_{31}^{2u})\\ &\quad-(\sigma_{12}^{2u}\vee\sigma_{22}^{2u}\vee\sigma_{32}^{2u}))^2.\end{aligned}$$

定理 3.1 若 $\lambda_1>0$, 且对于任意的初值 $(S(0),I(0),V(0))\in\mathbb{R}^3_+$, 系统 (2.3) 的解随机最终有界且随机持久.

证 定义 $V=N+\frac{1}{N}, N=S+I+V$, 对于 $X(t)\in\mathbb{R}^3_+$, 当 $|X(t)|\to\infty$ 时, 则 $V(X(t))\to\infty$. 根据 Itô 公式可得

$$\begin{aligned}
LV=&A(t)-\mu(t)N-\alpha(t)I-\frac{1}{N^2}(A(t)-\mu(t)N-\alpha(t)I)+\frac{1}{N^3}[S^2(\sigma_{11}(t)+\sigma_{12}(t)S)^2\\
&+I^2(\sigma_{21}(t)+\sigma_{22}(t)I)^2+V^2(\sigma_{31}(t)+\sigma_{32}(t)V)^2]\\
\le\ & A^u-\mu^lN-\frac{A^l}{N^2}+\frac{\mu^u}{N}+\frac{\alpha^u(S+I+V)}{N^2}+\frac{1}{N^3}\Big[(\sigma_{11}^{2u}\vee\sigma_{21}^{2u}\vee\sigma_{31}^{2u})N^2\\
&+(2\sigma_{11}^u\sigma_{12}^uS^2+2\sigma_{21}^u\sigma_{22}^uI^3+2\sigma_{31}^u\sigma_{32}^uV^3)N^3+(\sigma_{12}^{2u}\vee\sigma_{22}^{2u}\vee\sigma_{32}^{2u})N^4\Big]\\
\le\ & A^u-\mu^lN-\frac{A^l}{N^2}+\frac{\mu^u}{N}+\frac{\alpha^u}{N}+\frac{1}{N^3}\Big[(\sigma_{11}^{2u}\vee\sigma_{21}^{2u}\vee\sigma_{31}^{2u})N^2\\
&+2(\sigma_{11}^u\sigma_{12}^u\vee\sigma_{21}^u\sigma_{22}^u\vee\sigma_{31}^u\sigma_{32}^u)N^3+(\sigma_{12}^{2u}\vee\sigma_{22}^{2u}\vee\sigma_{32}^{2u})N^4\Big]\\
\le\ & A^u-\mu^lN-\frac{A^l}{N^2}+\frac{\mu^u}{N}+\frac{\alpha^u}{N}+\frac{(\sigma_{11}^{2u}\vee\sigma_{21}^{2u}\vee\sigma_{31}^{2u})}{N}+2(\sigma_{11}^u\sigma_{12}^u\vee\sigma_{21}^u\sigma_{22}^u\vee\sigma_{31}^u\sigma_{32}^u)\\
&+(\sigma_{12}^{2u}\vee\sigma_{22}^{2u}\vee\sigma_{32}^{2u})N\\
\le\ & -[\mu^l-(\sigma_{12}^{2u}\vee\sigma_{22}^{2u}\vee\sigma_{32}^{2u})](N+\frac{1}{N})+A^u+2(\sigma_{11}^u\sigma_{12}^u\vee\sigma_{21}^u\sigma_{22}^u\vee\sigma_{31}^u\sigma_{32}^u)-\frac{A^l}{N^2}\\
&+\frac{1}{N}\Big(\mu^l+\mu^u+\alpha^u+(\sigma_{11}^{2u}\vee\sigma_{21}^{2u}\vee\sigma_{31}^{2u})-(\sigma_{12}^{2u}\vee\sigma_{22}^{2u}\vee\sigma_{32}^{2u})\Big)\\
\le\ & -\lambda_1V+A^u+2(\sigma_{11}^u\sigma_{12}^u\vee\sigma_{21}^u\sigma_{22}^u\vee\sigma_{31}^u\sigma_{32}^u)-A^l\left(\frac{l}{N^2}-\frac{E}{NA^l}+\frac{E^2}{4A^{2l}}\right)+\frac{E^2}{4A^l}\\
\le\ & -\lambda_1V+A^u+2(\sigma_{11}^u\sigma_{12}^u\vee\sigma_{21}^u\sigma_{22}^u\vee\sigma_{31}^u\sigma_{32}^u)-A^l\left(\frac{l}{N}-\frac{E}{2A^l}\right)^2+\frac{E^2}{4A^l}\\
\le\ & -\lambda_1V+C,
\end{aligned}$$

其中 $E=(\mu^l+\mu^u+\alpha^u+(\sigma_{11}^{2u}\vee\sigma_{21}^{2u}\vee\sigma_{31}^{2u})-(\sigma_{12}^{2u}\vee\sigma_{22}^{2u}\vee\sigma_{32}^{2u}))$.

意味着

$$\begin{aligned}
\mathbb{E}\Big[e^{\lambda_1t}V(t)\Big]&=\mathbb{E}[V(0)]+\mathbb{E}\left[\int_0^te^{\lambda_1s}[\lambda_1V(s)+LV(s)]ds\right]\\
&\le\mathbb{E}[V(0)]+C\mathbb{E}\left[\int_0^te^{\lambda_1s}ds\right]\\
&\le\mathbb{E}[V(0)]+\frac{C}{\lambda_1}\Big[e^{\lambda_1t}-1\Big].
\end{aligned}$$

因此

$$\mathbb{E}[V(t)]\le e^{-\lambda_1t}\mathbb{E}[V(0)]+\frac{C}{\lambda_1}\Big[1-e^{\lambda_1t}\Big]\le\mathbb{E}[V(0)]+\frac{C}{\lambda_1}=:H.$$

选取足够大的常数 ς, 使得 $\frac{C}{\lambda_1\varsigma}<1$, 运用 Chebyshev's 不等式得

$$\mathbb{P}\left\{N+\frac{1}{N}>\varsigma\right\}\le\frac{1}{\varsigma}\mathbb{E}\left[N+\frac{1}{N}\right]\le\frac{1}{\varsigma}\left[e^{-\lambda_1t}\mathbb{E}[V(0)]+\frac{C}{\lambda_1}\Big[1-e^{\lambda_1t}\Big]\right]\le\frac{H}{\varsigma}.$$

意味着

$$\limsup_{t\to\infty}\mathbb{P}\left\{N+\frac{1}{N}>\varsigma\right\}\le\frac{1}{\varsigma}\limsup_{t\to\infty}\mathbb{E}\left[N+\frac{1}{N}\right]\le\frac{C}{\lambda_1\varsigma}=:\delta.$$

可得

$$\limsup_{t\to\infty}\mathbb{P}\{N>\varsigma\}\le\limsup_{t\to\infty}\mathbb{P}\left[N+\frac{1}{N}>\varsigma\right]\le\delta.$$

对于 $|X|\le N$, 有

$$\limsup_{t\to\infty}\mathbb{P}[|X|\le\varsigma]<\delta.$$

类似地

$$\liminf_{t\to\infty}\mathbb{P}\{N>\varsigma\}\le\delta,\quad \liminf_{t\to\infty}\mathbb{P}\left\{\frac{1}{N}>\varsigma\right\}\le\delta.$$

所以

$$\liminf_{t\to\infty}\mathbb{P}\{N\le\varsigma\}\ge1-\delta,\quad \liminf_{t\to\infty}\mathbb{P}\left\{\frac{N}{\sqrt{3}}\ge\frac{1}{\sqrt{3}\varsigma}\right\}\ge1-\delta.$$

注意到 $N\ge|X|\ge\frac{N}{\sqrt{3}}$, 则有

$$\liminf_{t\to\infty}\mathbb{P}\{|X|\le\varsigma\}\ge1-\delta,\quad \liminf_{t\to\infty}\mathbb{P}\left\{|X|\ge\frac{1}{\sqrt{3}\varsigma}\right\}\ge1-\delta.$$

定理 3.1 证毕. ∎

下面研究系统 (2.3) 周期解的存在性. 定义参数

$$R_0^s=\frac{\left\langle\sqrt{(1-q(t))A(t)\beta(t)}\right\rangle_T^2}{\left\langle\mu(t)+p(t)+\sigma_{11}^2(t)+\frac{2\sigma_{12}^{2u}(A(t)+(\gamma(t)+\varepsilon(t))\lambda_2)}{\sigma_{11}^l\sigma_{12}^l}\right\rangle_T\left\langle\mu(t)+\gamma(t)+\alpha(t)+\frac{1}{2}\sigma_{21}^2(t)\right\rangle_T}.$$

定理 3.2 当 $R_0^s>1$ 时，系统 (2.3) 至少存在一个非平凡正 T - 周期解.

证 根据定理 3.1 的证明可知，对于任意的 $0<\delta<1$, 存在 $\Omega_\delta\in\Omega$, 满足 $\mathbb{P}\{\Omega_\delta\}\ge1-\delta$, 并且对于所有的 $\omega\in\Omega_\delta$ 都有 $N<\frac{H}{\delta}=:\lambda_2$, 定义函数

$$V=K(V_1+\omega(t))+V_2+V_3,$$

其中 $V_1=-k_1\ln S-k_2\ln I+k_3(\sigma_{11}^l+\sigma_{12}^lS)^n, V_2=-\ln S-\ln V, V_3=\frac{1}{\theta}(S+I+V)^\theta$,

$$k_1=\frac{\langle(1-q)A\rangle_T}{\left\langle\mu(t)+p(t)+\sigma_{11}^2(t)+\frac{2\sigma_{12}^{2u}(A(t)+(\gamma(t)+\varepsilon(t))\lambda_2)}{\sigma_{11}^l\sigma_{12}^l}\right\rangle_T},$$

$$k_2=\frac{\langle(1-q)A\rangle_T}{\left\langle\mu(t)+\gamma(t)+\alpha(t)+\frac{1}{2}\sigma_{21}^2(t)\right\rangle_T},\quad k_3=\frac{2k_1\sigma_{12}^{2u}}{n(1-n)\sigma_{12}^{2l}(\sigma_{11}^l)^n},K,n,\theta$$

均为正常数，且 $n<1,\theta<1$. 运用 Itô 公式可得

$$\begin{aligned}LV_1=&-\frac{k_1}{S}\Big((1-q(t))A(t)-\beta(t)SI^m-(\mu(t)+p(t))S+\gamma(t)I+\varepsilon(t)V\Big)\\&-\frac{k_2}{I}\Big(\beta(t)SI^m-(\mu(t)+\gamma(t)+\alpha(t))I\Big)+\frac{k_1}{2}(\sigma_{11}(t)+\sigma_{12}(t)S)^2\\&+\frac{k_2}{2}(\sigma_{21}(t)+\sigma_{22}(t)I)^2+k_3n\sigma_{12}^l\Big[A(t)(\sigma_{11}^l+\sigma_{12}^lS)^{n-1}-q(t)A(t)(\sigma_{11}^l+\sigma_{12}^lS)^{n-1}\\&-\beta(t)SI^m(\sigma_{11}^l+\sigma_{12}^lS)^{n-1}-(\mu(t)+p(t))S(\sigma_{11}^l+\sigma_{12}^lS)^{n-1}+\gamma(t)I(\sigma_{11}^l+\sigma_{12}^lS)^{n-1}\\&+\varepsilon(t)V(\sigma_{11}^l+\sigma_{12}^lS)^{n-1}\Big]-\frac{k_3}{2}n(1-n)\sigma_{12}^{2l}S^2(\sigma_{11}^l+\sigma_{12}^lS)^{n-2}(\sigma_{11}(t)+\sigma_{12}(t)S)^2\end{aligned}$$

$$\begin{aligned}
&\leq -\frac{k_1}{S}((1-q(t))A(t)+k_1\Big(\mu(t)+p(t)+\sigma_{11}^2(t)+\frac{2\sigma_{12}^{2u}A(t)}{(1-n)\sigma_{11}^l\sigma_{12}^l}\Big)\\
&\quad+\frac{2k_1\sigma_{12}^{2u}}{(1-n)\sigma_{11}^l\sigma_{12}^l}(\gamma(t)I+\varepsilon(t)V)+k_2\Big(\mu(t)+\gamma(t)+\alpha(t)+\frac{1}{2}\sigma_{21}^2(t)\Big)\\
&\quad-k_2\beta(t)SI^{m-1}+\frac{k_2}{2}\sigma_{22}^2(t)I^2+k_2\sigma_{21}(t)\sigma_{22}(t)I+k_1\beta(t)I^m\\
&\leq -2\sqrt{k_1k_2(1-q(t))A(t)\beta(t)}+k_2\beta(t)S+k_1\Big(\mu(t)+p(t)+\sigma_{11}^2(t)\\
&\quad+\frac{2\sigma_{12}^{2u}(A(t)+(\gamma(t)+\varepsilon(t))\lambda_2)}{(1-n)\sigma_{11}^l\sigma_{12}^l}\Big)+k_2\Big(\mu(t)+\gamma(t)+\alpha(t)+\frac{1}{2}\sigma_{21}^2(t)\Big)\\
&\quad-k_2\beta(t)SI^{m-1}+\frac{k_2}{2}\sigma_{22}^2(t)I^2+k_2\sigma_{21}(t)\sigma_{22}(t)I+k_1\beta(t)I^m\\
&=:-R_0(n,t)+k_2\beta(t)S-k_2\beta(t)SI^{m-1}+\frac{k_2}{2}\sigma_{22}^2(t)I^2+k_2\sigma_{21}(t)\sigma_{22}(t)I+k_1\beta(t)I^m,
\end{aligned}$$

其中

$$\begin{aligned}
R_0(n,t)=2\sqrt{k_1k_2(1-q(t))A(t)\beta(t)}-k_1\Big(&\mu(t)+p(t)+\sigma_{11}^2(t)\\
&+\frac{2\sigma_{12}^{2u}(A(t)+(\gamma(t)+\varepsilon(t))\lambda_2)}{(1-n)\sigma_{11}^l\sigma_{12}^l}\Big)-k_2\Big(\mu(t)+\gamma(t)+\alpha(t)+\frac{1}{2}\sigma_{21}^2(t)\Big).
\end{aligned}$$

从而有

$$\begin{aligned}
L(V_1+\omega(t))\leq{}&-R_0(n,t)+k_2\beta(t)S-k_2\beta(t)SI^{m-1}+\frac{k_2}{2}\sigma_{22}^2(t)I^2\\
&+k_2\sigma_{21}(t)\sigma_{22}(t)I+k_1\beta(t)I^m+\omega'(t).
\end{aligned}$$

设 $\omega(t)$ 是 T - 周期函数并且满足 $\omega'(t)=R_0(t)-\langle R_0\rangle_T$, 选取足够小的 n 使 $R_0(n,t)>0$, 当 $n\to 0$ 时，可得

$$\begin{aligned}
L(V_1+\omega(t))&\leq -\langle R_0\rangle_T+k_2\beta^uS-k_2\beta^lSI^{m-1}+\frac{k_2}{2}\sigma_{22}^{2u}I^2+k_2\sigma_{21}(t)\sigma_{22}(t)I+k_1\beta^uI^m\\
&=-2\langle(1-q(t))A\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+k_2\beta^uS-k_2\beta^lSI^{m-1}+\frac{k_2}{2}\sigma_{22}^{2u}I^2\\
&\quad+k_2\sigma_{21}(t)\sigma_{22}(t)I+k_1\beta^uI^m.
\end{aligned}$$

类似地

$$\begin{aligned}
LV_2&=-\frac{1}{S}\Big[(1-q(t))A(t)-\beta(t)SI^m-(\mu(t)+p(t))S+\gamma(t)I+\varepsilon(t)V\Big]\\
&\quad-\frac{1}{V}\Big[q(t)A(t)+p(t)S-(\mu(t)+\varepsilon(t))V\Big]+\frac{1}{2}(\sigma_{11}(t)+\sigma_{12}(t)S)^2\\
&\quad+\frac{1}{2}(\sigma_{31}(t)+\sigma_{32}(t)V)^2\\
&\leq -\frac{(1-q)A}{S}+\beta I^m-p\frac{S}{V}+\sigma_{12}^{2u}S^2+\sigma_{32}^{2u}V^2+2\mu^u+p^u+\varepsilon^u+\sigma_{11}^{2u}+\sigma_{31}^{2u},\\
LV_3&=(S+I+V)^{\theta-1}(A(t)-\mu(t)(S+I+V)-\alpha(t)I)-\frac{1}{2}(1-\theta)(S+I+V)^{\theta-2}\\
&\quad\times\Big[S^2(\sigma_{11}(t)+\sigma_{12}(t)S)^2+I^2(\sigma_{21}(t)+\sigma_{22}(t)I)^2+V^2(\sigma_{31}(t)+\sigma_{32}(t)V)^2\Big]\\
&\leq (S+I+V)^{\theta-1}(A(t)-\mu(t)(S+I+V)-\alpha(t)I)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}(1-\theta)(S+I+V)^{\theta-2}\Big(\sigma_{12}^2S^4+\sigma_{22}^2I^4+\sigma_{32}^2V^4\Big)\\
\le&-\frac{1}{54}(1-\theta)\Theta_2(S+I+V)^{\theta+2}-\mu^l(S+I+V)^\theta+\frac{A^u}{S^{1-\theta}},
\end{aligned}
$$

其中 $\Theta_2=\sigma_{12}^{2l}\wedge\sigma_{22}^{2l}\wedge\sigma_{32}^{2l}$. 基于上述分析，可得

$$
\begin{aligned}
LV\le&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+Kk_2\beta^uS-Kk_2\beta^lSI^{m-1}+\frac{Kk_2}{2}\sigma_{22}^{2u}I^2\\
&+Kk_2\sigma_{21}\sigma_{22}I+Kk_1\beta^uI^m-\frac{(1-q)A}{S}+\beta I^m-p\frac{S}{V}+\sigma_{12}^{2u}S^2+\sigma_{32}^{2u}V^2+2\mu^u+p^u\\
&+\varepsilon^u+\sigma_{11}^{2u}+\sigma_{31}^{2u}-\frac{1}{54}(1-\theta)\Theta_2(S+I+V)^{\theta+2}-\mu^l(S+I+V)^\theta+\frac{A^u}{S^{1-\theta}}\\
=&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+F+Kk_2\beta^uS-Kk_2\beta^lSI^{m-1}+\frac{Kk_2}{2}\sigma_{22}^{2u}I^2\\
&+Kk_2\sigma_{21}\sigma_{22}I+Kk_1\beta^uI^m-p\frac{S}{V}-\frac{(1-q)A}{S}+\beta I^m+\sigma_{12}^{2u}S^2+\sigma_{32}^{2u}V^2\\
&-\frac{1}{54}(1-\theta)\Theta_2(S+I+V)^{\theta+2}-\mu^l(S+I+V)^\theta+\frac{A^u}{S^{1-\theta}}\\
\le&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+\frac{Kk_2}{2}\sigma_{22}^{2u}I^2+Kk_2\beta^u\lambda_2+Kk_2\sigma_{21}^u\sigma_{22}^uI\\
&-Kk_2\beta^lSI^{m-1}+Kk_1\beta^uI^m+F-\frac{(1-q^u)A^l}{S}+\beta^uI^m+\sigma_{12}^{2u}S^2+\sigma_{32}^{2u}V^2\\
&-\frac{1}{54}(1-\theta)\Theta_2S^{\theta+2}-\frac{1}{54}(1-\theta)\Theta_2I^{\theta+2}-\frac{1}{54}(1-\theta)\Theta_2V^{\theta+2}-p^l\frac{S}{V}\\
&-\mu^l(S+I+V)^\theta+\frac{A^u}{S^{1-\theta}}\\
\le&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+\frac{Kk_2}{2}\sigma_{22}^{2u}I^2+Kk_2\beta^u\lambda_2+Kk_2\sigma_{21}^u\sigma_{22}^uI\\
&-Kk_2\beta^lSI^{m-1}+Kk_1\beta^u\lambda_2^{m-1}I+F-\frac{(1-q^u)A^l}{S}+\beta^u\lambda_2^{m-1}I+\sigma_{12}^{2u}S^2+\sigma_{32}^{2u}V^2\\
&-\frac{1}{54}(1-\theta)\Theta_2S^{\theta+2}-\frac{1}{54}(1-\theta)\Theta_2I^{\theta+2}-\frac{1}{54}(1-\theta)\Theta_2V^{\theta+2}\\
&-p^l\frac{S}{V}-\mu^l(S+I+V)^\theta+\frac{A^u}{S^{1-\theta}}\\
=:&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+F+f_1(I)+f_2(S)+f_3(V)+g(S)\\
&-\mu^l(S+I+V)^\theta-Kk_2\beta^lSI^{m-1}-p^l\frac{S}{V},
\end{aligned}
$$

其中

$$
f_1(I)=-\frac{1-\theta}{54}\Theta_2I^{\theta+2}+\frac{Kk_2}{2}\sigma_{22}^{2u}I^2+K(k_2\sigma_{21}^u\sigma_{22}^u+k_1\beta^u\lambda_2^{m-1})I+Kk_2\beta^u\lambda_2+\beta^u\lambda_2^{m-1}I,
$$

$$
f_2(S)=-\frac{1-\theta}{54}\Theta_2S^{\theta+2}+\sigma_{12}^{2u}S^2,\quad f_3(V)=-\frac{1-\theta}{54}\Theta_2V^{\theta+2}+\sigma_{32}^{2u}V^2,
$$

$$
g(S)=-\frac{(1-q^u)A^l}{S}+\frac{A^u}{S^{1-\theta}}=\frac{A^u}{S^{1-\theta}}\Big(-\frac{(1-q^u)A^l}{A^uS^\theta}+1\Big),F=2\mu^u+p^u+\varepsilon^u+\sigma_{11}^{2u}+\sigma_{31}^{2u}.
$$

选取足够大的正常数 K 使得

$$
-2K\langle(1-q)A\rangle_T((R_0^s)^{\frac{1}{2}}-1)+F+f_2^u+f_3^u+g^u\le-2.
$$

定义有界区域

$$D=\Big\{(S,I,V)\in\mathbb{R}^3_+,\varepsilon_1\le S\le\frac{1}{\varepsilon_1},\varepsilon_1\le I\le\frac{1}{\varepsilon_1},\varepsilon_2\le V\le\frac{1}{\varepsilon_2}\Big\}.$$

选择足够小的正常数 ε_1 和 $\varepsilon_2,\varepsilon_2=\varepsilon_1^2$, 并且满足如下不等式

$$\begin{aligned}
&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+g^u+f_1(\varepsilon_1)+f_2^u+f_3^u\le-1,\\
&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+g^u+f_1^u+f_2^u+f_3^u-\frac{(1-q^u)A^l}{\varepsilon_1}+\frac{A^u}{\varepsilon_1^{1-\theta}}\le-1,\\
&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+g^u+f_1^u+f_2^u+f_3^u-p^l\frac{1}{\varepsilon_1}\le-1,\\
&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+g^u+f_1\Big(\frac{1}{\varepsilon_1}\Big)+f_2^u+f_3^u\le-1,\\
&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+f_1^u+f_2\Big(\frac{1}{\varepsilon_1}\Big)+f_3^u+g^u\le-1,\\
&-2K\langle(1-q)A\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+f_1^u+f_2^u+f_3\Big(\frac{1}{\varepsilon_1}\Big)+g^u\le-1.
\end{aligned}$$

为了讨论方便，将 $\mathbb{R}^3_+\setminus D$ 分成以下 6 个区域

$$D_1=\{(S,I,V)\in\mathbb{R}^3_+:0<I<\varepsilon_1\},\quad D_2=\{(S,I,V)\in\mathbb{R}^3_+:0<S<\varepsilon_1\},$$

$$D_3=\{(S,I,V)\in\mathbb{R}^3_+:\varepsilon_1<S,\varepsilon_1<I,0<V<\varepsilon_2\},\quad D_4=\Big\{(S,I,V)\in\mathbb{R}^3_+:\frac{1}{\varepsilon_1}<I\Big\},$$

$$D_5=\Big\{(S,I,V)\in\mathbb{R}^3_+:\frac{1}{\varepsilon_1}<S\Big\},\quad D_6=\Big\{(S,I,V)\in\mathbb{R}^3_+:\frac{1}{\varepsilon_2}<V\Big\}.$$

显然， $D^c=\bigcup\limits_{i=1}^{6}D_i$. 下面将分 6 种情况在 D^c 证明 $LV\le-1$.

1) 当 $(S,I,V)\in[0,\infty)\times D_1$, 有

$$\begin{aligned}
LV&\le-2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+f_1(I)+f_2(S)+f_3(V)+g(S)\\
&\le-2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+g^u+f_1(\varepsilon_1)+f_2^u+f_3^u.
\end{aligned}$$

2) 当 $(S,I,V)\in[0,\infty)\times D_2$, 有

$$\begin{aligned}
LV&\le-2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+f_1(I)+f_2(S)+f_3(V)+g(S)\\
&\le-2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+g^u+f_1^u+f_2^u+f_3^u-\frac{(1-q^u)A^l}{\varepsilon_1}+\frac{A^u}{\varepsilon_1^{1-\theta}}.
\end{aligned}$$

3) 当 $(S,I,V)\in[0,\infty)\times D_3$, 有

$$\begin{aligned}
LV&\le-2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+f_1(I)+f_2(S)+f_3(V)+g(S)\\
&\le-2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+g^u+f_1^u+f_2^u+f_3^u-p^l\frac{1}{\varepsilon_1}.
\end{aligned}$$

4) 当 $(S,I,V)\in[0,\infty)\times D_4$, 有

$$\begin{aligned}
LV&\le-2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+f_1(I)+f_2(S)+f_3(V)+g(S)\\
&\le-2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac12}-1\big)+F+g^u+f_1(\frac{1}{\varepsilon_1})+f_2^u+f_3^u.
\end{aligned}$$

5) 当 $(S,I,V)\in[0,\infty)\times D_5$, 有

$$\begin{aligned}LV&\le -2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+F+f_1(I)+f_2(S)+f_3(V)+g(S)\\&\le -2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+F+f_1^u+f_2(\frac{1}{\varepsilon_1})+f_3^u+g^u.\end{aligned}$$

6) 当 $(S,I,V)\in[0,\infty)\times D_6$, 有

$$\begin{aligned}LV&\le -2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+F+f_1(I)+f_2(S)+f_3(V)+g(S)\\&\le -2K\big\langle(1-q)A\big\rangle_T\big((R_0^s)^{\frac{1}{2}}-1\big)+F+f_1^u+f_2^u+f_3(\frac{1}{\varepsilon_1})+g^u.\end{aligned}$$

综上，对任意的 $(S,I,V)\in[0,\infty)\times D^c$, 有 $LV\le -1$. ▍

4 平均意义下的持久性

定理 4.1 当 $R_1^\star=\frac{\hat\beta(1-\check q)\hat A}{\lambda_2(\check\mu+\check p)(\check\mu+\check\gamma+\check\alpha+\frac{1}{2}\check\sigma_{21}^2+\frac{1}{2}\check\sigma_{22}^2\lambda_2^2)}>1, k\in\hbar$ 时，系统 (2.2) 的染病者在平均意义下具有持久性.

证 根据定理 3.1 可知，对于任意的 $0<\delta<1$, 存在 $\Omega_\delta\in\Omega$, 满足 $\mathbb{P}\{\Omega_\delta\}\ge 1-\delta$, 并且对于所有的 $\omega\in\Omega_\delta$ 都有 $N<\frac{H}{\delta}=:\lambda_2$. 利用 Itô 公式可得

$$\begin{aligned}\mathrm{d}(S+I)\ge{}&(1-\check q)\hat A-(\check\mu+\check p)S-(\check\mu+\check\alpha)I+S(\sigma_{11}(k)+\sigma_{12}(k)S)\mathrm{d}B_1(s)\\&+I(\sigma_{21}(k)+\sigma_{22}(k)I)\mathrm{d}B_2(s).\end{aligned}$$

将上式两端从 0 到 t 积分得

$$\begin{aligned}\phi(t)=:{}&\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}\\&\ge(1-\check q)\hat A-(\check\mu+\check p)\langle S\rangle_t-(\check\mu+\check\alpha)\langle I\rangle_t+\frac{N_1}{t}+\frac{N_2}{t},\end{aligned}$$

其中 $N_1=\int_0^t S(\sigma_{11}(k)+\sigma_{12}(k)S)\mathrm{d}B_1(s)$, $N_2=\int_0^t I(\sigma_{21}(k)+\sigma_{22}(k)I)\mathrm{d}B_2(s)$.

由 $\lim\limits_{t\to\infty}\phi(t)=0$ 可得

$$\langle S\rangle_t\ge\frac{1}{\check\mu+\check p}\Big[(1-\check q)\hat A-\phi(t)-(\check\mu+\check\alpha)\langle I\rangle_t+\frac{N_1}{t}+\frac{N_2}{t}\Big],$$

运用 Itô 公式可得

$$\mathrm{d}\ln I\ge\Big[\frac{\hat\beta S}{\lambda_2}-(\check\mu+\check\gamma+\check\alpha)-\frac{1}{2}\check\sigma_{12}^2-\check\sigma_{21}\check\sigma_{22}I-\frac{1}{2}\check\sigma_{22}^2\lambda_2^2\Big]\mathrm{d}t+(\sigma_{21}(k)+\sigma_{22}(k)I)\mathrm{d}B_2(t).$$

将上式从 0 到 t 积分得

$$\begin{aligned}\frac{\ln I(t)-\ln I(0)}{t}&\ge\frac{\hat\beta}{\lambda_2}\langle S\rangle_t-(\check\mu+\check\gamma+\check\alpha)-\frac{1}{2}\check\sigma_{21}^2-\check\sigma_{21}\check\sigma_{22}\langle I\rangle_t+\frac{N_3}{t}-\frac{1}{2}\check\sigma_{22}^2\lambda_2^2\\&\ge\frac{\hat\beta}{(\check\mu+\check p)\lambda_2}\Big[(1-\check q)\hat A-\phi(t)-(\check\mu+\check\alpha)\langle I\rangle_t+\frac{N_1}{t}+\frac{N_2}{t}\Big]\\&\quad+\frac{N_3}{t}-(\check\mu+\check\gamma+\check\alpha)-\frac{1}{2}\check\sigma_{21}^2-\check\sigma_{21}\check\sigma_{22}\langle I\rangle_t-\frac{1}{2}\check\sigma_{22}^2\lambda_2^2,\end{aligned}$$

其中 $N_3=\int_0^t(\sigma_{21}(k)+\sigma_{22}(k)I)\mathrm{d}B_2(s)$. 由强大数定理可知

$$\lim_{t\to\infty}\frac{N_1}{t}=\lim_{t\to\infty}\frac{N_2}{t}=\lim_{t\to\infty}\frac{N_3}{t}=0,\lim_{t\to\infty}\frac{\ln I(t)-\ln I(0)}{t}\le 0.$$

从而得

$$\begin{aligned}\liminf_{t\to\infty}\langle I\rangle_t\ge&\ \frac{1}{\frac{\hat{\beta}(\check{\mu}+\check{\alpha})}{(\check{\mu}+\check{p})\lambda_2}+\check{\sigma}_{21}\check{\sigma}_{22}}\left(\frac{\hat{\beta}\lambda_2^{m-1}(1-\check{q})\hat{A}(k)}{(\check{\mu}+\check{p})\lambda_2}-(\check{\mu}+\check{\gamma}+\check{\alpha})-\frac{1}{2}\check{\sigma}_{21}^2-\frac{1}{2}\check{\sigma}_{22}^2\lambda_2^2\right)\\ =&\ \frac{1}{\frac{\hat{\beta}(\check{\mu}+\check{\alpha})}{(\check{\mu}+\check{p})\lambda_2}+\check{\sigma}_{21}\check{\sigma}_{22}}\left(\check{\mu}+\check{\gamma}+\check{\alpha}+\frac{1}{2}\check{\sigma}_{21}^2+\frac{1}{2}\check{\sigma}_{22}^2\lambda_2^2\right)\\ &\cdot\left(\frac{\hat{\beta}(1-\check{q})\hat{A}}{\lambda_2(\check{\mu}+\check{p})(\check{\mu}+\check{\gamma}+\check{\alpha}+\frac{1}{2}\check{\sigma}_{21}^2+\frac{1}{2}\check{\sigma}_{22}^2\lambda_2^2)}-1\right)\\ =:&\ \frac{1}{\frac{\hat{\beta}(\check{\mu}+\check{\alpha})}{(\check{\mu}+\check{p})\lambda_2}+\check{\sigma}_{21}\check{\sigma}_{22}}\left(\check{\mu}+\check{\gamma}+\check{\alpha}+\frac{1}{2}\check{\sigma}_{21}^2+\frac{1}{2}\check{\sigma}_{22}^2\lambda_2^2\right)(R_1^\star-1)>0.\end{aligned}$$

则系统 (2.2) 的染病者在平均意义下具有持久性. ▍

5 遍历平稳分布

本节讨论系统 (2.2) 的遍历平稳分布的存在性. 为此，定义参数

$$\begin{aligned}R_0^\star=&\left\{\left(\sum_{k=1}^N\pi_k\sqrt{(1-q(k))A(k)\beta(k)}\right)^2\right\}\Big/\left\{\sum_{k=1}^N\pi_k\Big(\mu(k)+p(k)+\sigma_{11}^2(k)\right.\\ &\left.+\frac{2\check{\sigma}_{12}^2(A(k)+(\gamma(k)+\varepsilon(k))\lambda_2)}{\hat{\sigma}_{11}\hat{\sigma}_{12}}\Big)\sum_{k=1}^N\pi_k\left(\mu(k)+\gamma(k)+\alpha(k)+\frac{1}{2}\sigma_{21}^2(k)\right)\right\}.\end{aligned}$$

定理 5.1 当 $R_0^\star>1$ 时，对于任意初值 $(S(0),I(0),V(0),r(0))\in\mathbb{R}_+^3\times\hbar$, 系统 (2.2) 的解 $(S(t),I(t),V(t),r(t))$ 存在唯一的遍历平稳分布.

证 对 $i\ne j,\gamma_{ij}>0$ 成立，其扰动矩阵为

$$\begin{aligned}A=&\ (a_{ij}(S,I,V,K))\\ =&\begin{pmatrix}(S(\sigma_{11}(k)+\sigma_{12}(k)S))^2 & 0 & 0\\ 0 & (I(\sigma_{21}(k)+\sigma_{22}(k)I))^2 & 0\\ 0 & 0 & (V(\sigma_{31}(k)+\sigma_{32}(k)V))^2\end{pmatrix}.\end{aligned}$$

为了使引理 2.2 中的条件 iii) 成立，考虑有界开集 $D=[\varepsilon,\frac{1}{\varepsilon}]\times[\varepsilon,\frac{1}{\varepsilon}]\times[\varepsilon,\frac{1}{\varepsilon}]$, 其中 $\varepsilon>0$ 是充分小的正常数. 对任意 $(S,I,V,k)\in D^c(D^c=\mathbb{R}_+^3\setminus D)\times\hbar,\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}_+^3$, 有

$$\begin{aligned}\sum_{i,j=1}^3a_{ij}(S,I,V,k)\xi_i\xi_j=&\ (\sigma_{11}(k)S+\sigma_{12}(k)S^2)^2\xi_1^2+(\sigma_{21}(k)I+\sigma_{22}(k)I^2)^2\xi_2^2\\ &+(\sigma_{31}(k)V+\sigma_{32}(k)V^2)^2\xi_3^2\\ \ge&\ M_1\|\xi\|^2,\end{aligned}$$

其中

$$M_1=\min_{(S,I,V,k)\in D^c\times\hbar}\Big\{(\sigma_{11}(k)S+\sigma_{12}(k)S^2)^2,(\sigma_{21}(k)I+\sigma_{22}(k)I^2)^2,(\sigma_{31}(k)V+\sigma_{32}(k)V^2)^2\Big\}.$$

显然引理 2.2 中的条件 ii) 成立．在 $\mathbb{R}_+^3\times\hbar$ 上定义函数

$$V=M(V_1+\omega(k))+V_2+V_3,$$

其中

$$V_1=-c_1\ln S-c_2\ln I+c_3(\hat\sigma_{11}(k)+\hat\sigma_{12}(k)S)^n,V_2=-\ln S-\ln V,V_3=\frac{1}{\theta}(S+I+V)^\theta,$$

$$c_1=\frac{\sum\limits_{k=1}^N\pi_k(1-q(k))A(k)}{\sum\limits_{k=1}^N\pi_k\Big(\mu(k)+p(k)+\sigma_{11}^2(k)+\frac{2\check\sigma_{12}^2(A(k)+(\gamma(k)+\varepsilon(k))\lambda_2)}{(1-n)\hat\sigma_{11}\hat\sigma_{12}}\Big)},$$

$$c_2=\frac{\sum\limits_{k=1}^N\pi_k(1-q(k))A(k)}{\sum\limits_{k=1}^N\pi_k\big(\mu(k)+\gamma(k)+\alpha(k)+\frac12\sigma_{21}^2(k)\big)},\quad c_3=\frac{2c_1\check\sigma_{12}^2}{n(1-n)\hat\sigma_{12}^2\hat\sigma_{11}^n},$$

$0<n<1,0<\theta<1,M$ 为正常数．根据 Itô 公式得

$$\begin{aligned}L(V_1+\omega(k))\le&-\frac{c_1}{S}(1-q(k))A(k)\\&+c_1\Big(\mu(k)+p(k)+\sigma_{11}^2(k)+\frac{2\check\sigma_{12}^2(A(k)+(\gamma(k)+\varepsilon(k))\lambda_2)}{(1-n)\hat\sigma_{11}\hat\sigma_{12}}\Big)\\&+c_2\Big(\mu(k)+\gamma(k)+\alpha(k)+\frac12\sigma_{21}^2(k)\Big)-c_2\beta(k)SI^{m-1}+\frac{c_2}{2}\sigma_{22}^2(k)I^2\\&+c_1\beta(k)I^m+c_2\sigma_{21}(k)\sigma_{22}(k)I+\sum_{l\in\hbar}\gamma_{kl}\omega(l)\\\le&-2\sqrt{c_1c_2(1-q(k))A(k)\beta(k)}+c_2\beta(k)S+c_1\Big(\mu(k)+p(k)+\sigma_{11}^2(k)\\&+\frac{2\check\sigma_{12}^2(A(k)+(\gamma(k)+\varepsilon(k))\lambda_2)}{(1-n)\hat\sigma_{11}\hat\sigma_{12}}\Big)+c_2\Big(\mu(k)+\gamma(k)+\alpha(k)+\frac12\sigma_{21}^2(k)\Big)\\&-c_2\beta(k)SI^{m-1}+\frac{c_2}{2}\sigma_{22}^2(k)I^2+c_2\sigma_{21}(k)\sigma_{22}(k)I+c_1\beta(k)I^m+\sum_{l\in\hbar}\gamma_{kl}\omega(l)\\=:&-\bar R_0(n,k)+\frac{c_2}{2}\sigma_{22}^2(k)I^2+c_2\beta(k)S+c_2\sigma_{21}(k)\sigma_{22}(k)I-c_2\beta(k)SI^{m-1}\\&+c_1\beta(k)I^m+\sum_{l\in\hbar}\gamma_{kl}\omega(l),\end{aligned}$$

其中

$$\begin{aligned}\bar R_0(n,k)=&2\sqrt{c_1c_2(1-q(k))A(k)\beta(k)}-c_2\Big(\mu(k)+\gamma(k)+\alpha(k)+\frac12\sigma_{21}^2(k)\Big)\\&-c_1\Big(\mu(k)+p(k)+\sigma_{11}^2(k)+\frac{2\check\sigma_{12}^2(A(k)+(\gamma(k)+\varepsilon(k))\lambda_2)}{(1-n)\hat\sigma_{11}\hat\sigma_{12}}\Big).\end{aligned}$$

选取足够小的 n, 使得 $\bar{R}_0(n,k)>0$, 当 $n\to 0^+$ 时，有

$$\begin{aligned}L(V_1+\omega(k)) &\le -\bar{R}_0(n,k)+\frac{c_2}{2}\check{\sigma}_{22}^2I^2+c_2\check{\beta}S+c_2\check{\sigma}_{21}\check{\sigma}_{22}I-c_2\hat{\beta}SI^{m-1}+c_1\check{\beta}I^m+\sum_{l\in\hbar}\check{\gamma}_{kl}\omega(l)\\ &\to -\bar{R}_0(k)+\frac{c_2}{2}\check{\sigma}_{22}^2I^2+c_2\check{\beta}S+c_2\check{\sigma}_{21}\check{\sigma}_{22}I-c_2\hat{\beta}SI^{m-1}+c_1\check{\beta}I^m+\sum_{l\in\hbar}\check{\gamma}_{kl}\omega(l).\end{aligned}$$

为了给出 $\omega(l)$, 定义向量 $\bar{R}_0=(\bar{R}_0(1),\bar{R}_0(2),\cdots,\bar{R}_0(N))$. 由于生成矩阵 Γ 不可约，对于任意的 $\bar{R}_0(k)$, 存在集合 $\omega=(\omega(1),\omega(2),\cdots,\omega(N))$, 满足泊松系统 [16]: $\Gamma\omega-\bar{R}_0(k)=-\sum\limits_{h=1}^{N}\pi_h\bar{R}_0(h)$. 相应地可得

$$\sum_{l\in\hbar}\gamma_{kl}\omega(l)-\bar{R}_0(k)=-\sum_{h=1}^{N}\pi_h\bar{R}_0(h).$$

综上可得

$$\begin{aligned}L(V_1+\omega(k)) \le{}& -2\sum_{k=1}^{N}\pi_k(1-\check{q})\hat{A}\Big[(R_0^{\star})^{\frac{1}{2}}-1\Big]+c_2\check{\beta}S+\frac{c_2}{2}\check{\sigma}_{22}^2I^2-c_2\hat{\beta}SI^{m-1}\\ &+c_2\check{\sigma}_{21}\check{\sigma}_{22}I+c_1\check{\beta}I^m.\end{aligned}$$

运用类似定理 3.2 的方法可得 $LV\le -1$. 由引理 2.2 可知，系统 (2.2) 的解具有唯一的遍历分布. ▮

6 系统的灭绝性

在本节中，给出随机系统 (2.2) 的染病者灭绝的条件. 定义参数

$$R_1^s=\frac{\check{\beta}\lambda_2^{m-1}}{(\hat{\mu}+\hat{\gamma}+\hat{\alpha}+\frac{1}{2}\hat{\sigma}_{21}^2)}.$$

定理 6.1 当 $R_1^s<1$ 时，对于任意的初值 $(S(0),I(0),V(0),r(0))\in\mathbb{R}_+^3\times\hbar$, 系统 (2.2) 的染病者是灭绝的.

证 根据 Itô 公式有

$$\begin{aligned}\mathrm{d}[\ln I(t)] &= \left[\frac{\beta(k)SI^m-(\mu(k)+\gamma(k)+\alpha(k))I}{I}-\frac{1}{2}(\sigma_{21}(k)+\sigma_{22}(k)I)^2\right]\mathrm{d}t\\ &\quad+(\sigma_{21}(k)+\sigma_{22}(k)I)\mathrm{d}B_2(t)\\ &\le \left[\check{\beta}SI^{m-1}-(\hat{\mu}+\hat{\gamma}+\hat{\alpha})-\frac{1}{2}(\sigma_{21}^2(k)+\sigma_{22}^2(k)I^2)\right]\mathrm{d}t\\ &\quad+(\sigma_{21}(k)+\sigma_{22}(k)I)\mathrm{d}B_2(t).\end{aligned}$$

两边从 0 到 t 积分，可得

$$\begin{aligned}\ln I &\le \check{\beta}\int_0^t S\lambda_2^{m-1}\mathrm{d}s-\int_0^t(\hat{\mu}+\hat{\gamma}+\hat{\alpha})\mathrm{d}s-\frac{1}{2}\int_0^t(\sigma_{21}^2(k)+\sigma_{22}^2(k)I^2)\mathrm{d}s\\ &\quad+\int_0^t\sigma_{21}(k)\mathrm{d}B_2(s)+\int_0^t\sigma_{22}(k)I\mathrm{d}B_2(s)+\ln(I(0))\\ &=:\check{\beta}\int_0^t S\lambda_2^{m-1}\mathrm{d}s-\int_0^t(\hat{\mu}+\hat{\gamma}+\hat{\alpha})\mathrm{d}s-\frac{1}{2}\int_0^t(\sigma_{21}^2(k)+\sigma_{22}^2(k)I^2)\mathrm{d}s\\ &\quad+M_1(t)+M_2(t)+\ln(I(0)),\end{aligned}$$

其中 $M_1(t)=\int_0^t\sigma_{22}(k)I\mathrm{d}B_2(s)$, $M_2(t)=\int_0^t\sigma_{21}(k)\mathrm{d}B_2(s)$.

通过指数鞅不等式，对于任何正常数 T,ϑ,υ, 有

$$\mathbb{P}\Big\{\sup_{0<t\leq T}[M_1(t)-\frac{\vartheta}{2}\langle M_1,M_1\rangle_t]>\upsilon\Big\}\leq \mathrm{e}^{-\vartheta\upsilon},$$

其中 $\langle M_1,M_1\rangle_t=\int_0^t\sigma_{22}^2(k)I^2\mathrm{d}s$, 令 $T=Q,\vartheta=1,\upsilon=2\ln Q$. 可得

$$\mathbb{P}\Big\{\sup_{0<t\leq T}[M_1(t)-\frac{1}{2}\langle M_1,M_1\rangle_t]>2\ln Q\Big\}\leq \frac{1}{Q^2}.$$

因此对于几乎所有的 $\omega\in\Omega$ 存在随机整数 $Q_0\in Q_0(\omega)$, 使得对于 $Q\geq Q_0+M$ (其中 M 为足够大的正常数), 有

$$\sup_{0<t\leq T}\left[M_1-\frac{1}{2}\langle M_1,M_1\rangle_t\right]\leq 2\ln Q.$$

对于所有的 $0\leq t\leq Q, Q\geq Q_0$, a.s. 有

$$M_1(t)\leq 2\ln Q+\frac{1}{2}\langle M_1,M_1\rangle_t=2\ln Q+\frac{1}{2}\int_0^t\sigma_{22}^2I^2\mathrm{d}s.$$

不难看出，对于 $0\leq Q-1\leq t\leq Q$, 有

$$\begin{aligned}\frac{\ln I}{t}\leq&\frac{\check{\beta}}{t}\int_0^t S\lambda_2^{m-1}\mathrm{d}s-\frac{1}{t}\int_0^t\Big(\hat{\mu}+\hat{\gamma}+\hat{\alpha}+\frac{1}{2}\sigma_{21}^2(k)\Big)\mathrm{d}s-\frac{1}{2t}\int_0^t\sigma_{22}^2(k)I^2\mathrm{d}s\\&+\frac{\ln(I(0))}{t}+\frac{2\ln Q}{t}+\frac{1}{2t}\int_0^t\sigma_{22}^2(k)I^2\mathrm{d}s+\frac{M_2(t)}{t}\\\leq&\frac{\check{\beta}}{t}\int_0^t S\lambda_2^{m-1}\mathrm{d}s-\frac{1}{t}\int_0^t\Big(\hat{\mu}+\hat{\gamma}+\hat{\alpha}+\frac{1}{2}\sigma_{21}^2(k)\Big)\mathrm{d}s+\frac{\ln(I(0))}{t}+\frac{2\ln Q}{t}+\frac{M_2(t)}{t},\end{aligned}$$

根据强大数定理得 $\lim\limits_{t\to\infty}\frac{M_2(t)}{t}=0$. 因此

$$\lim_{t\to\infty}\frac{\ln I}{t}\leq\check{\beta}\lambda_2^{m-1}-(\hat{\mu}+\hat{\gamma}+\hat{\alpha}+\frac{1}{2}\hat{\sigma}_{21}^2)<0.$$

从而有 $\lim\limits_{t\to\infty}\ln I(t)=0$. 定理得证. ∎

7 数值模拟

本节进行数值模拟，为此我们使用如下离散方程

$$\begin{cases}S_{n+1}=S_n+\Big((1-q_n)A_n-\beta_nS_nI_n^m-(\mu_n+p_n)S_n+\gamma_nI_n+\varepsilon_nV_n\Big)\Delta t\\\qquad\quad+S_n(\sigma_{11,n}+\sigma_{12,n}S_n)\Delta B_{1,n},\\I_{n+1}=I_n+(\beta_nS_nI_n^m-(\mu_n+\gamma_n+\alpha_n)I_n)\Delta t+I_n(\sigma_{21,n}+\sigma_{22,n}I_n)\Delta B_{2,n},\\V_{n+1}=V_n+(q_nA_n+p_nS_n-(\mu_n+\varepsilon_n)V_n)\Delta t+V_n(\sigma_{31,n}+\sigma_{32,n}V_n)\Delta B_{3,n}.\end{cases}$$

选取以下参数值

$$A=1.5+0.5\sin t,\beta=5.7+0.2\sin t,\mu=0.2+0.1\sin t,\gamma=0.4+0.3\sin t,$$
$$p=0.6+0.1\sin t,q=0.2+0.01\sin t,\varepsilon=0.35+0.02\sin t,\alpha=0.3+0.2\sin t,$$
$$\sigma_{11}=0.5+0.1\sin t,\sigma_{12}=0.4+0.1\sin t,\sigma_{21}=0.2+0.01\sin t,$$
$$\sigma_{22}=0.1+0.02\sin t,\sigma_{31}=0.05+0.01\sin t,\sigma_{32}=0.06+0.01\sin t.$$

经计算可得 $R_0^s = 4.99 > 1$, 由定理 3.2 可知系统 (2.3) 存在非平凡正周期解 (见图 1).

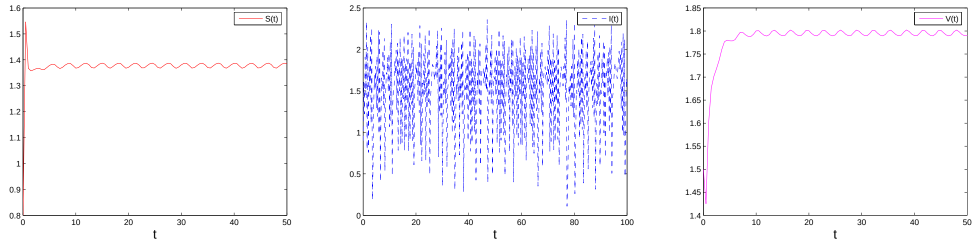

图 1 系统 (2.3) 的初值为 $(0.8, 0.9, 1.5)$ 的解曲线, $S(t), I(t), V(t)$ 呈周期性波动

在系统 (2.2) 中, 我们考虑右连续的马尔科夫链 $r(t)$ 是取值在 $\hbar = \{1, 2\}$ 上的, $r(t)$ 的稳态分布为 $\pi = (\pi_1, \pi_2) = (\frac{5}{7}, \frac{2}{7})$. 选取以下参数值.

(1) 当 $r(t) = 1$ 时, $A = 2, \beta = 3, \mu = 0.65$, $\gamma = 0.2$, $p = 0.5, q = 0.2$, $\varepsilon = 0.25$, $\alpha = 0.45, \sigma_{11} = 0.1, \sigma_{12} = 0.01, \sigma_{21} = 0.05, \sigma_{22} = 0.01, \sigma_{31} = 0.05, \sigma_{32} = 0.01$.

(2) 当 $r(t) = 2$ 时, $A = 2, \beta = 4, \mu = 0.5, \gamma = 0.35, p = 0.7, q = 0.15, \varepsilon = 0.35$, $\alpha = 0.5, \sigma_{11} = 0.1, \sigma_{12} = 0.01, \sigma_{21} = 0.05, \sigma_{22} = 0.02, \sigma_{31} = 0.05, \sigma_{32} = 0.02$.

经计算可得 $R_0^\star = 1.25 > 1$, 根据定理 5.1 可知系统 (2.2) 存在唯一的遍历平稳分布 (见图 2).

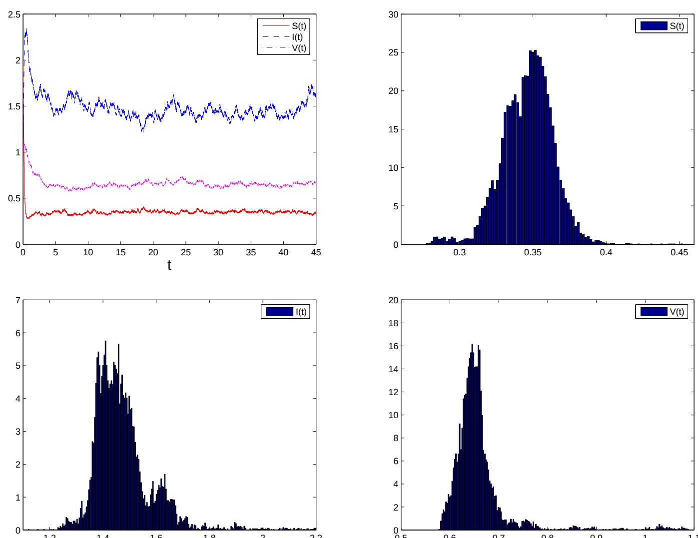

图 2 系统 (2.3) 的初值为 $(2, 1, 1)$ 的解曲线及解的密度函数

选取以下参数值.

(1) 当 $r(t) = 1$ 时, $A = 0.8, \beta = 0.2, \mu = 0.9, \gamma = 0.9, p = 0.35$, $q = 0.75, \varepsilon = 0.9$, $\alpha = 0.3, \sigma_{11} = 0.7, \sigma_{12} = 0.7, \sigma_{21} = 0.7, \sigma_{22} = 0.7, \sigma_{31} = 0.7, \sigma_{32} = 0.7$.

(2) 当 $r(t) = 2$ 时, $A = 0.81, \beta = 0.21, \mu = 0.89, \gamma = 0.85, p = 0.36, q = 0.65, \varepsilon = 0.9$, $\alpha = 0.3, \sigma_{11} = 0.7, \sigma_{12} = 0.7, \sigma_{21} = 0.7, \sigma_{22} = 0.7, \sigma_{31} = 0.7, \sigma_{32} = 0.7$.

经计算可得 $R_1^s = 0.78 < 1$, 故根据定理 6.1 可知系统的感染者趋于灭绝 (见图 3).

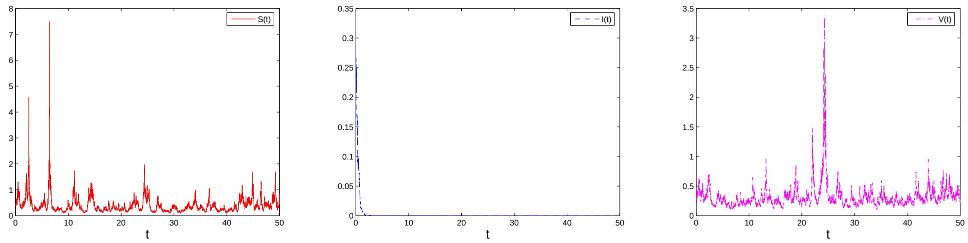

图 3　系统 (2.3) 的初值为 $(0.4,0.3,0.3)$ 的解曲线，此时 $S(t),I(t)$ 是持久的， $V(t)$ 是灭绝的

8　结论

本文考虑环境之间的随机切换建立了一类具有非线性随机扰动非自治 SIVS 模型．首先忽略有色噪声的影响，定义了模型的阈值，证明了随机正周期解的存在性，并进行数值模拟验证了理论结果．表明易感人群，感染人群及接种疫苗后获得免疫的人群的数量由于季节的变化、天气的转变等因素的影响呈周期性变化．

在有色噪声的影响下，我们证明了模型的平均持久性、遍历性和绝灭性等．这些数学结论的生物意义如下．

由定理 4.1 知当 $R_1^\star>1$ 时系统 (2.2) 在平均意义下是持久的，表明传染病在种群内将长期存在；

由定理 5.1 知当 $R_0^\star>1$ 时系统 (2.2) 的随机过程 $(S(t),I(t),V(t),r(t))$ 是遍历的且具有唯一的平稳分布，表明虽然易感者、感染者及接种者的数量随着时间的推移而波动，但它们的平均值收敛于固定值；

由定理 6.1 知当 $R_1^s<1$ 时系统 (2.2) 是绝灭的，表明感染者的数量随着时间的推移最终趋于 0, 疾病消失．

参 考 文 献

[1] Li J, Ma Z. Qualitative analyses of SIS epidemic model with vaccination and varying total population size. Mathematical and Computer Modelling, 2002, **35**(11/12): 1235–1243

[2] Farnoosh R, Parsamanesh M. Disease extinction and persistence in a discrete-time SIS epidemic model with vaccination and varying population size. Filomat, 2017, **31**(15): 4735–4747

[3] Li J, Ma Z. Global analysis of SIS epidemic models with variable total population size. Mathematical and Computer Modelling, 2004, **39**(11/12): 1231–1242

[4] Kribs-Zaleta C, Velasco-Hernández J. A simple vaccination model with multiple endemic states. Mathematical Biosciences, 2000, **164**(2): 183–201

[5] Gumel A, Moghadas S. A qualitative study of a vaccination model with nonlinear incidence. Applied Mathematics and Computation, 2015, **143**(2): 409–419

[6] Li X, Wang J, Ghosh M. Stability and bifurcation of an SIVS epidemic model with treatment and age of vaccination. Applied Mathematical Modelling, 2010, **34**(2): 437–450

[7] Knipl D, Pilarczyk P, Rost G. Rich bifurcation structure in a two-patch vaccination model. Siam Journal on Applied Dynamical Systems, 2015, **14**(2): 980–1017

[8] Nie L, Shen J, Yang C. Dynamic behavior analysis of SIVS epidemic models with state-dependent pulse vaccination. Nonlinear Analysis: Hybrid Systems, 2018, **27**: 258–270

[9] Zhao Y, Jiang D, O'Regan D. The extinction and persistence of the stochastic SIS epidemic model with vaccination. Physical A: Statistical Mechanics and Its Applications, 2013, **392**(20): 4916–4927

[10] Liu Q, Jiang D, Shi N, et al. The threshold of a stochastic SIS epidemic model with imperfect vaccination. Mathematics and Computers in Simulation, 2018, **144**: 78–90

[11] Zhao Y, Jiang D. Dynamics of stochastically perturbed SIS epidemic model with vaccination. Abstract and Applied Analysis, 2013, (2013): 1401–1429
[12] Zhao Y, Jiang D. The threshold of a stochastic SIS epidemic model with vaccination. Applied Mathematics and Computation, 2014, **243**: 718–727
[13] Zhang W, Meng X, Dong Y. Periodic solution and ergodic stationary distribution of stochastic SIRI epidemic systems with nonlinear perturbations. Journal of Systems Science and Complexity, 2019, **32**(4): 1104–1124
[14] 林青腾．随机干扰下若干传染病模型的动力学行为 [D]. 福州：福州大学， 2018
Lin Q T. Dynamics Behavior of a Number of Infectious Disease Model Under Random Disturbance[D]. Fuzhou: Fuzhou University, 2018
[15] Takeuchi Y, Du N H, Hieu N T, Sato K. Evolution of predator-prey systems described by a Lotka-Volterra equation under random environment. Journal of Mathematical Analysis and Applications, 2017, **323**(2): 938–957
[16] Zu L, Jiang D, O'Regan D. Conditions for persistence and ergodicity of a stochastic Lotka-Volterra predator-prey model with regime switching. Communications in Nonlinear Science and Numerical Simulation, 2015, **29**(1–3): 1–11
[17] Zhang X, Wang K. Stochastic SIR model with jumps. Applied Mathematics Letters, 2013, **26**(8): 867–874
[18] 胡俊娜．转换机制下的具有非线性发生率的随机传染病模型的动力学性质 [D]. 乌鲁木齐：新疆大学， 2018
Hu J N. Dynamic Properties of Random Epidemic Model with Nonlinear Incidence Under the Transformation Mechanism[D]. Urumqi: Xinjiang university, 2018
[19] Has'miniskii R. Stochastic Stability of Differential Equations[M]. Alphenaan den Rijn: Sijthoff and Noordhff, 1980
[20] Zhu C, Yin G. Asymptotic properties of hybrid diffusion systems. SIAM Journal on Control and Optimization, 2007, **46**(4): 1155–1179
[21] Jiang D, Shi N, Li X. Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation. Journal of Mathematical Analysis and Applications, 2008, **340**(1): 588–597

Qualitative Analysis of a Stochastic SIVS Epidemic Model with Nonlinear Perturbations Under Regime Switching

Zhang Zhonghua Zhang Qian
(*School of Sciences, Xi'an University of Science and Technology, Xi'an 710054*)

Abstract: In this paper, we present a stochastic SIVS epidemic model with nonlinear perturbations under regime switching. For the non-autonomous stochastic SIVS epidemic system with white noise, we provide results regarding the stochastic boundedness, stochastic permanence in mean, and we prove that the system has at least one nontrivial positive T-periodic solution by using Lyapunov function and Hasminskii's theory. For the system with Markov conversion, we establish sufficient conditions for existence of ergodic stationary distribution, and the thresholds for persistence in mean and the extinction of infected persons was obtained, respectively. Finally, some numerical simulations are carried out to support the theoretical results.

Key words: Stochastic SIVS epidemic model; Nonlinear perturbation; Markov chain; Nonlinear incidence.

MR(2010) Subject Classification: 60H30